Name: Solutions

BC Calculus

TOPICS
Techniques of Integration

Multiple Choice

C

1. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sqrt{1-\cos x}}\,dx =$

   (A) - 2

   (B) $\frac{2}{3}$

   (C) 2

   (D) $\frac{1}{2}$

   (E) none of these

$u = 1-\cos x$
$du = \sin x\,dx$
$\int_0^1 u^{-1/2}\,du = 2u^{1/2}\Big|_0^1 = 2(1-0) = 2$

D

2. If $f(x) = \displaystyle\int_0^x \frac{1}{\sqrt{t^3+2}}\,dt$, then which of the following is false? $f(x) > 0$

   (A) $f(0) = 0$ T

   (B) $f(1) > 0$ T

   (C) $f'(1) = \frac{1}{\sqrt{3}}$ T

   (D) $f(-1) > 0$ F

   (E) f is continuous at x for all $x \geq 0$.

$f(0) = \int_0^0 = 0$ T
$f(1) = \int_0^1 > 0$
$f'(x) = \frac{1}{\sqrt{x^3+2}} \Rightarrow f'(1) = \frac{1}{\sqrt{1+2}}$
$f(-1) = \int_0^{-1} = -\int_{-1}^0 < 0$

B

3. If the substitution $u = \sqrt{x+1}$ is used, then $\displaystyle\int_0^3 \frac{dx}{x\sqrt{x+1}}$ is equivalent to

   (A) $\displaystyle\int_1^2 \frac{du}{u^2-1}$

   (B) $\displaystyle\int_1^2 \frac{2\cdot du}{u^2-1}$

   (C) $2\displaystyle\int_0^3 \frac{du}{u^2-1}$

   (D) $2\displaystyle\int_1^2 \frac{du}{u(u^2-1)}$

   (E) $2\displaystyle\int_0^3 \frac{du}{u(u-1)}$

$\int$

$u^2 - 1 = x$
$u^2 = x+1$
$du = \frac{1}{2\sqrt{x+1}}\,dx$

$\boxed{2du = \frac{dx}{\sqrt{x+1}}}$

$\int_1^2 \frac{2du}{u^2-1}$

$2\int_1^2 \frac{du}{u^2-1}$

E

4. $\int \sin x \cos x \, dx =$

$\sin x \cos x$

$2 \sin x \cos x - \sin x \cos x$

$\frac{2}{2} \sin x \cos x$

$\frac{d}{dx}$

I. $\dfrac{\sin^2 x}{2} + C$ ✓

II. $\sin^2 x - \int \sin x \cos x \, dx$ ✓

III. $-\cos^2 x - \int \sin x \cos x \, dx$

$+2\cos x \sin x$

$-\sin x \cos x$

$\sin x \cos x$

$u =$

(A) I only
(B) I and II
(C) I and III
(D) II and III
(E) I, II, and III

C

5. $\displaystyle\int_0^1 \frac{e^x}{e^x+1} \, dx =$

$u = e^x + 1$

$du = e^x dx$

$\displaystyle\int_2^{e+1} \frac{du}{u} = \ln(e+1) - \ln 2 = \ln \frac{e+1}{2}$

(A) $1+e$
(B) e
(C) $\ln \dfrac{e+1}{2}$
(D) $\ln 2$
(E) - $\ln 2$

C

6. $\displaystyle\int_0^2 \sqrt{4-x^2} \, dx =$

quarter ~~semi~~ circle of radius = 2

(A) $\dfrac{8}{3}$

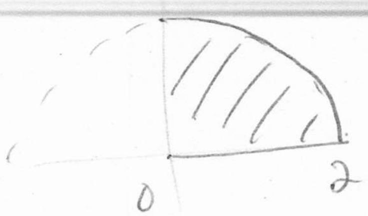
(B) $\dfrac{16}{3}$

(C) pi

(D) $2pi$

(E) $4p\,i$

$\dfrac{\pi(2)^2}{4} = \pi$



7. If $\int_1^4 f(x)\,dx = 6$, what is the value of $\int_1^4 f(5-x)\,dx$? $= -\int_4^1 f(u)\,du$

let $u = 5-x$

$du = -dx$ $= \int_1^4 f(u)\,du$

$-du = dx$

A

(A) 6

(B) 3

(C) 0

(D) - 1

(E) - 6

B

8. A table of integrals includes $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$. It follows that $\int_0^1 \frac{dx}{\sqrt{4-x^2}} =$

(A) $\frac{pi}{3}$

(B) $\frac{pi}{6}$

(C) $\frac{pi}{12}$

(D) $2(\sqrt{3} - 2)$

(E) $2 - \sqrt{3}$

$a=2$

$u=x$

$= \arcsin\frac{x}{2}\Big|_0^1$

$= \arcsin\frac{1}{2} - \cancel{\arcsin 0}^{\,0}$

$= \frac{\pi}{6}$

A

9. $\int xe^{2x}\,dx =$ $\frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx$

$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$

(A) $\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$

(B) $\frac{xe^{2x}}{2} - \frac{e^{2x}}{2} + C$

(C) $\frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + C$

(D) $\frac{xe^{2x}}{2} + \frac{e^{2x}}{2} + C$

(E) $\frac{x^2e^{2x}}{4} + C$

$u = x$ $du = dx$

$dv = e^{2x}dx$ $v = \frac{e^{2x}}{2}$

Solutions

1. Find each of the following antiderivatives. Carefully show your work so that—in the event that you struggle—I can give you partial credit.

$du = dx \quad v = \sin x$
$u = x \quad dv = \cos x\,dx$

$u^2 = x + 4$
$u = \sqrt{x+4} \quad u^2 - 4 = x$
$du = \frac{1}{2\sqrt{x+4}}dx$
$2\sqrt{x+4}\,du = dx$

(a) $\int \frac{x^2}{x^3+27}\,dx$ (b) $\int x\cos x\,dx$ (c) $\int x\sqrt{x+4}\,dx$

$u = x^3 + 27$
$du = 3x^2\,dx$
$\frac{du}{3} = x^2\,dx$

$\frac{1}{3}\int \frac{du}{u}$

$\frac{1}{3}\ln|u| + C$

$\boxed{\frac{1}{3}\ln|x^3+27| + C}$

$x\sin x - \int \sin x\,dx$

$x\sin x + \int -\sin x\,dx$

$\boxed{x\sin x + \cos x + C}$

$\int (u^2-4)\,2\underbrace{\sqrt{x+4}}_{u}\,du$

$2\int (u^2-4)(\underbrace{x+4}_{u^2})\,du$

$2\int (u^4 - 4u^2)\,du$

$2\left(\frac{u^5}{5} - \frac{4u^3}{3}\right) + C$

$\boxed{\frac{2}{5}(x+4)^{5/2} - \frac{4}{3}(x+4)^{3/2} + C}$

2. Find each of the following antiderivatives. Carefully show your work.

(a) $\int \cos(x)\sin(2x)\,dx$ (b) $\int e^{2x}\sin x\,dx$ (c) $\int \frac{\sqrt{1-x^2}}{x}\,dx$

Note: In solving part (c), you will need to remember that

$\int \sec\theta\,d\theta = \ln\left|\sec\theta + \tan\theta\right|\,d\theta + C$

a) $\sin(2x) = 2\sin x\cos x$

$\int \cos(x)\,2\sin(x)\cos(x)\,dx$

$2\int (\cos x)^2\sin(x)\,dx$

$u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

$2\int u^2\,du$

$-\frac{2}{3}u^3 + C$

$\boxed{-\frac{2}{3}\cos^3 x + C}$

(b) $\int e^{2x}\sin x\,dx = \frac{e^{2x}\sin x}{2} - \int \frac{e^{2x}}{2}\cos x\,dx$

let
$u = \sin x \quad v = \frac{e^{2x}}{2}$
$du = \cos x\,dx \quad dv = e^{2x}$

let
$w = \cos x \quad V = \frac{e^{2x}}{2}$
$dw = -\sin x\,dx \quad dv = e^{2x}$

$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{2}\sin x - \frac{1}{2}\left(\frac{e^{2x}\cos x}{2} + \int \frac{e^{2x}\sin x}{2}\right)$

$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{2}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x$

$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{e^{2x}}{2}\sin x - \frac{1}{4}e^{2x}\cos x + C$

$\int e^{2x}\sin x\,dx = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + C$

3. Find each of the following antiderivatives. Carefully show your work.

(a) $\int x \arctan x \, dx$ (b) $\int x^7 \cos(x^4)\, dx$

$u = x^4$

$du = 4x^3 dx$

$\frac{du}{4} = x^3 dx$

Hint: In solving part (b), you should try a u-substitution first.

(a) $u = \arctan x$ $dv = x\,dx$

$du = \frac{1}{1+x^2} dx$ $v = \frac{x^2}{2} dx$

$= \frac{x^2}{2} \arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2} dx$

$\int x^4 \cdot x^3 \cos(x^4) dx$

$\frac{1}{4}\int u \cdot \cos(u) du$

let $w = u$ $dv = \cos u$

$dw = du$ $v = \sin u$

$\frac{1}{4}\left(w \sin u - \int \sin u \, du\right)$

$\frac{1}{4} w \sin u + \cos u + C$

$\boxed{\frac{1}{4}\left(x^4 \sin(x^4) + \cos(x^4)\right) + C}$

4. Let f be a continuous and differentiable function for all real numbers. Show that the following definite integrals have the same numerical value.

$$2\int_{\ln\frac{\sqrt{2}}{2}}^{0} e^{2x} \cdot f(e^{2x})\, dx \qquad \int_0^{\frac{\pi}{3}} \sin x \cdot f(\cos x)\, dx \qquad \int_{\sqrt{e}}^{e} \frac{f(\ln x)}{x} dx$$

$u = e^{2x}$

$du = 2e^{2x} dx$

$2\ln\frac{\sqrt{2}}{2}$

e

$e^{\ln\frac{1}{2}} = \frac{1}{2}$

$\boxed{\int_{1/2}^{1} f(u)\, du}$

$u = \cos x$

$du = -\sin x\, dx$

$u(0) = \cos(0) = 1$

$u(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = 1/2$

$-\int_1^{1/2} f(u)\, du$

$\boxed{\int_{1/2}^{1} f(u) du}$

$u = \ln x$

$du = \frac{1}{x} dx$

$\boxed{\int_{1/2}^{1} f(u) du}$

Solutions

BC Calculus | **TOPICS** Techniques of Integration

1. $\int \frac{1}{x^2+x}\,dx = \int \frac{1}{x(x+1)} = \int \left(\frac{A}{x} + \frac{B}{x+1}\right) dx$

D

(A) $\frac{1}{2}\arctan\left(x+\frac{1}{2}\right)+C$

(B) $\ln\left|x^2+x\right|+C$

(C) $\ln\left|\frac{x+1}{x}\right|+C$

(D) $\ln\left|\frac{x}{x+1}\right|+C$

(E) none of these

$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$1 = A(x+1) + Bx$

let $x=0$: $A=1$

let $x=-1$: $-B=1$, $B=-1$

$= \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$\ln|x| - \ln|x+1| + C$

$= \ln\left|\frac{x}{x+1}\right| + C$

2. $\int_2^3 \frac{8}{x^2+x-2}\,dx = \int_2^3 \left(\frac{A}{x+2} + \frac{B}{x-1}\right) dx$

D

(A) $\frac{-33}{20}$

(B) $\frac{-9}{20}$

(C) $\frac{8}{3}\ln\frac{5}{2}$

(D) $\frac{8}{3}\ln\frac{8}{5}$

(E) $\frac{3}{8}\ln\frac{2}{5}$

$8 = A(x-1) + B(x+2)$

let $x=1$: $B=\frac{8}{3}$

let $x=2$: $A=\frac{-8}{3}$

$\int \frac{-8/3}{x+2} + \frac{8/3}{x-1}$

$-\frac{8}{3}\ln|x+2| + \frac{8}{3}\ln|x-1| \Big|_2^3$

$-\frac{8}{3}\left(\ln\left|\frac{x+2}{x-1}\right|\right)\Big|_2^3$

$-\frac{8}{3}\left(\ln\left(\frac{5}{2}\right) - \ln(4)\right)$

$-\frac{8}{3}(\ln 5 - \ln 2 - 2\ln 2)$

$-\frac{8}{3}(\ln 5 - 3\ln 2)$

$-\frac{8}{3}(\ln 5 - \ln 8)$

$-\frac{8}{3}\left(\ln\frac{5}{8}\right)$

$\frac{8}{3}\ln\frac{8}{5}$

3. Find the antiderivative. Carefully show your work.

$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \left(\frac{A}{(x-3)} + \frac{B}{(x+1)}\right) dx$

$5x-3 = A(x+1) + B(x-3)$

let $x=-1$: $-5-3 = -4B \Rightarrow B=2$

let $x=3$: $5(3)-3 = 4A \Rightarrow A=3$

$= \int \left(\frac{3}{x-3} + \frac{2}{x+1}\right) dx$

$= 3\ln|x-3| + 2\ln|x+1| + C$