HW: Page 130 #23, 30-33, 35, 37

DO NOW:

Any questions from last night's HW?

12 $g(x)=e^{x^2}$ $g'(x)=2xe^{x^2}$

$g(x)=1+x^2+\frac{x^4}{2!}+\cdots+\frac{x^{2n}}{n!}+\cdots$

$2xe^{x^2}=2x+\frac{4x^3}{2!}+\cdots+\frac{(2n)x^{2n-1}}{n!}+\cdots$

$2xe^{x^2}=2x+2x^3$

19 $C_n=\frac{f^{(n)}(a)}{n!}$

$f^{(n)}(0)=\frac{(-1)^n\cdot(n+1)}{n^2}$

$C_n=\frac{(-1)^n\cdot(n+1)}{n^2\cdot n!}$

$f(x)=3-2x+\frac{3}{8}x^2+\cdots+\frac{(-1)^n(n+1)}{n^2\cdot n!}x^n+\cdots$

$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(n+2)x^{n+1}}{(n+1)^2\cdot(n+1)!}\cdot\frac{n^2\cdot n!}{(n+1)x^n}\right|$

$=|x|\frac{n^2(n+2)}{(n+1)^{\cancel{3}4}}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|x|\cdot 0=0<1$

All real #'s

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To summarize our understanding of power series so far…

1. Every power series has an interval of convergence.

These are the x-values for which the series converges and for which we can get sensible values out of the infinite series.

2. Every power series has a radius of convergence.

The radius of convergence can be 0 (the power series converges only at its center), infinite (the power series converges for all *x*), or a positive finite number. In the latter case, the interval of convergence is an interval symmetric about its center (give or take the endpoints). The width of the interval is twice the radius of convergence.

3. The power series always converges absolutely in the interior of the interval of convergence.

This is usually determined by the ratio test, but if the power series is geometric, then we can use the geometric series test. To determine whether the series converges (conditionally or absolutely) or diverges at its endpoints, we use the convergence tests that we have seen in this chapter.

4. We can integrate or differentiate a power series term by term to produce a new power series.

The new power series will have the same radius of convergence (and center, of course), but its behavior at the endpoints may change. When we differentiate, we might lose endpoints from the interval of convergence. Conversely, when we integrate we may gain endpoints. There is no general way to predict what will happen; you just have to apply convergence tests to the endpoints of the new series.

Applications of Taylor Series: Integrals, Limits, and Sums Page 119

Okay, so why do we care? I care about Taylor series because I think they are really, really cool. You are free to disagree, but if you do I will have to present some more concrete examples of their utility.

The first has to do with filling a major hole in our ability to do calculus. Your calculus course has been mainly about finding and using derivatives and antiderivatives. Derivatives are no problem. Between the definition of the derivative and the various differentiation rules, you should be able to quickly write down the derivative of any differentiable function you meet. Antidifferentiation is trickier; many functions simply do not have explicit antiderivatives. We cannot integrate them. Taylor series provide us with a way for circumventing this obstacle, at least to a certain extent.

Example 1

Evaluate $\int e^{x^2}\,dx$.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\int e^{x^2}dx = 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{2n}}{n!} + \cdots$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \cdots + \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

Example 2

Compute $\int_0^1 e^{x^2}\,dx = \left[x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \cdots + \frac{x^{2n+1}}{(2n+1)\cdot n!} + \cdots\right]_0^1$

$$= \left(1 + \frac{1}{3} + \frac{1}{10} + \cdots + \frac{1}{(2n+1)\cdot n!} + \cdots\right)$$

$$- \left(0 + \frac{0}{3} + \frac{0}{10} + \cdots + \frac{0}{(2n+1)\cdot n!} + \cdots\right)$$

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Example 3

Use Maclaurin series to help evaluate $\lim\limits_{x\to 0}\frac{\sin x}{x}$ and $\lim\limits_{x\to 0}\frac{1-\cos x}{x}$

$$\lim_{x\to 0}\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots}{x}=$$

$$\lim_{x\to 0} 1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots=1$$

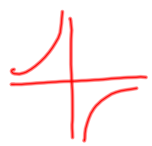
$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{\cancel{1}-(\cancel{1}+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots)}{x}=$$

$$\lim_{x\to 0}\frac{x}{2!}-\frac{x^3}{4!}+\cdots=0$$

Practice 2

Use a Maclaurin series to help evaluate $\lim\limits_{x\to 0}\frac{1-e^x}{x^2}$ $\lim\limits_{x\to 0}\frac{-e^x}{2x}$ d.n.e

$$\lim_{x\to 0}\frac{\cancel{1}-(\cancel{1}+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)}{x^2}=$$



$$\lim_{x\to 0}-\frac{1}{x}-\frac{1}{2}-\frac{x}{3!}+\frac{x^2}{4!}+\cdots$$ d.n.e.

Practice 1: p. 120

Use Taylor Series to compute

$$\int \sin(x^3)\,dx$$

Write 1st 3 terms and general term.

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$$

$$\int \sin x^3\,dx=\int x^3-\frac{x^9}{3!}+\frac{x^{15}}{5!}+\cdots+\frac{(-1)^n x^{6n+3}}{(2n+1)!}+\cdots dx$$

$$=\frac{x^4}{4}-\frac{x^{10}}{10\cdot 3!}+\frac{x^{16}}{16\cdot 5!}+\cdots+\frac{(-1)^n x^{6n+4}}{(6n+4)(2n+1)!}+\cdots$$

$$\int_0^1 \sin x^3\,dx\approx\frac{1}{4}-\frac{1}{60}+\frac{1}{16\cdot 120}$$ error is less than next term $\frac{1}{22\cdot 7!}$