HW: Page 129 #1,3,4,5,10, 12bcd, 19, 21
Due Friday

DO NOW:

Questions from HW?

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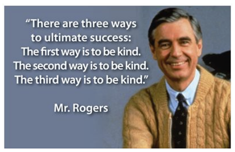

#68 The function f is defined by a power series as follows: $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{2n+1}$.

a. Find the interval of convergence of this series.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{2(n+1)+1}\cdot\frac{2n+1}{x^n}\right| = |x|\cdot\frac{2n+1}{2n+3}$$

$$\lim_{n\to\infty}|x|\frac{2n+1}{2n+3} = |x| \qquad |x|<1 \qquad 1 = \text{Radius}$$

Check $x=-1$ $\sum\frac{(-1)^n}{2n+1}$ conv by AST

Check $x=1$ $\sum\frac{1}{2n+1}$ diverges limit comp w/ $\frac{1}{n}$

$$-1\le x<1$$

b. Approximate $f(-1)$ by using a fourth-degree Maclaurin polynomial for f.

$$f(-1)=\frac{(-1)^0}{2(0)+1}+\frac{(-1)^1}{2(1)+1}+\dots+\frac{(-1)^4}{2(4)+1}$$

c. Estimate the error in your approximation from part (a) and give bounds for the value of $f(-1)$.

less than 1st omitted term

$$\frac{1}{2(5)+1}=\frac{1}{11}$$

Speed to ∞

Logs	Polynomial	Exponential	Factorial
Sloth	Turtle	Hare	Cheetah

$$\lim_{n\to\infty}\frac{n!}{10{,}000^n}=\infty$$

(11) $\sum\frac{(-1)^n\cdot n!}{10^n}$ n^{th} term test, div.

(15) $\sum\frac{(-3)^n}{n!+n^2}$ terms go to zero quickly

(45) $\sum\frac{n!(x-6)^n}{n^2+5}$

~~62~~ 65 Let $f(x)=\sum_{n=0}^{\infty} x^n$. $= 1 + x + x^2 + x^3 + \cdots$

a. Find the interval of convergence of this power series.

$|x| < 1$

$x = 1 \quad 1 + 1 + 1 + \cdots$ Go to the movies

$x = -1 \quad 1 - 1 + 1 - \cdots$

b. Express $f'(x)$ as a power series and find its interval of convergence.

$f(x) = 1 + x + x^2 + x^3 + \cdots$

$f'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad \sum_{n=1}^{\infty} n x^{n-1}$

$|x| < 1$

$x = 1 \quad 1 + 2 + 3 +$

$x = -1 \quad 1 - 2 + 3 + \cdots$ See you at the Cinema

c. Express $\int_0^x f(t)\,dt$ as a power series and find its interval of convergence.

$\int_0^x 1 + t + t^2 + t^3 + t^4\,dt = \quad \sum_{n=1}^{\infty} \frac{x^n}{n}$

$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

$|x| < 1$

$x = 1 \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} +$

$x = -1 \quad 1 - \frac{1}{2} + \frac{1}{3} + \cdots$ No ticket for you

$-1 \leq x < 1$

Questions about Taylor Polynomials from Section 2:

1. *Is there a systematic way to come up with the coefficients of a Taylor polynomial for a given function?*
2. *Can we know how big the error will be from using a Taylor polynomial?*
3. *When can we extend the interval on which the Taylor polynomial is a good fit indefinitely?*
4. *Can we match a function perfectly if we use infinitely many terms? Would that be meaningful?*

ANSWERS:

1) Yes. The rule is: The coefficient for the n^{th}-degree term should be $\frac{f^{(n)}(a)}{n!}$

2) Yes; we have learned one of two ways, the Alternating Series Error Bound, which works some of the time. ~~We may look at the other test~~, the Lagrange Error Bound, ~~later~~. This is harder to use, but works more often.

As you might expect, the answer to Question 3 has everything to do with interval of convergence, though we need to talk about Question 4 before we can appropriately frame Question 3 and its answer.

Would it be meaningful to have infinitely many terms?—has been answered already. A power series is a "polynomial" with infinitely many terms.

The answer to the first part—Can we match a function perfectly with a power series?—is: Yes!

The name we give a power series that represents a given function is a Taylor series (or a Maclaurin series if the center is 0).

Let's look back at our friend, the sine function.
Here is the Maclaurin (Taylor) Polynomial:

$$P_{2n+1}(x) = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \cdots + (-1)^n \cdot \tfrac{x^{2n+1}}{(2n+1)!}$$

So the Maclaurin (Taylor) Series will be:

$$x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \cdots + (-1)^n \cdot \tfrac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \cdot \tfrac{x^{2n+1}}{(2n+1)!} = \sin(x)$$

First three terms and the general term

This power series is an exact representation of the sine function.
It is not an approximation. It is the function.

Find the interval of convergence for this series

$$\lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| = \lim_{n\to\infty} x^2 \frac{1}{(2n+2)(2n+3)} = 0$$

All real #'s

It can be shown (but we will not) that this series does in fact converge to sin(x) for all real values of x.

We can use algebraic and calculus manipulation to get Taylor and Maclaurin series for other functions. Consider the Taylor polynomial

$\frac{1}{1-t} \approx 1 + t + t^2 + t^3$ which we figured out by considering a

geometric series with first term 1 and ratio *t*. $\int \frac{1}{1-x}\,dx = -\ln|1-x|$

Now we must turn to the question of why a Taylor polynomial for $f(x) = \frac{1}{1-x}$ is interesting. Well, it isn't. But we can integrate it, and that *will* be interesting. If we know, for example, that

$$\frac{1}{1-t} \approx 1 + t + t^2 + t^3,$$

then it should follow that

$$\int_0^x \frac{1}{1-t}\,dt \approx \int_0^x \left(1 + t + t^2 + t^3\right) dt.$$

t is just a dummy variable here. Carrying out the integration gives the following.

$$-\ln(1-t)\Big|_0^x \approx \left(t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4}\right)\Bigg|_0^x$$

$$-\ln(1-x) \approx x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4$$

$$\ln(1-x) \approx -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4$$

From here he we generalize.

$$\ln(1-x) \approx -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \cdots - \tfrac{1}{n}x^n$$

We can use other manipulations to get the following formulas from page 41. Let's see what those manipulations are.

$\frac{1}{1-x} = 1+x+x^2+x^3+\dots$

$\frac{1}{1-(-x)} = \frac{1}{1+x} \approx 1 - x + x^2 - x^3 + \dots + (-1)^n x^n$

$\frac{1}{1+(x^2)}$ $\quad \frac{1}{1+x^2} \approx 1 - x^2 + x^4 - \dots + (-1)^n x^{2n}$

$\ln(1-x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n}$

$\ln(1-(-x))$ $\quad \ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n+1} \cdot \frac{x^n}{n}$

$\arctan(x) \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$

$\int \frac{1}{1+x^2}dx$

$\arctan(1) = \frac{\pi}{4}$

$\pi \approx 4 \cdot \arctan(1)$

What about our other friend the Maclaurin series for f(x)=ln(1+x). The Maclaurin Polynomial for this function is:

$$P_n(x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \dots + (-1)^{n+1} \cdot \tfrac{x^n}{n}$$

so the Maclaurin Series will be:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n+1} \cdot \frac{x^n}{n} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$$

Find the **interval** of convergence for this series

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| =$ $\quad -1 < x \le 1$

$\lim_{n\to\infty} |x| \cdot \frac{n}{n+1} = |x| < 1$

$x = 1 \quad +1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ conv

$x = -1 \quad -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$ div

Definition:	If a function f is differentiable infinitely many times in some interval around $x = a$, then the Taylor series centered at a for f is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. If $a = 0$, then we can call the series a Maclaurin series.

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As you can see, a Taylor series is a special case of a power series where the coefficients c_n are determined by the function in question and the chosen center: $c_n = \frac{f^{(n)}(a)}{n!}$.

Though it is not part of the definition, we have seen that the Taylor series for a function is an alternative representation of the function that (in the nice cases that we care about) is perfectly faithful to how we understand the function to behave. This is something that is still amazing to me; by analyzing the derivatives of f at a single point, we are able to build a representation of f that is valid anywhere within in the interval of convergence.

It is as if every point on the curve has all the information resting inside of it to create the entire function. The DNA (the values of the derivatives) is there in every cell (x-value) of the organism (the function).