HW: Series Packet Page 77 #13 - 17, 20, 21, 23, 25
Page 77 #1 - 11 (odd)

Do Now:

Consider the geometric series

$$1 + x + x^2 + x^3 + x^4 + \cdots$$

1) What is the ratio? x

2) For what values of x does it converge?

$$|x| < 1$$

3) When it converges, what is the sum?

$$\frac{1}{1-x}$$

4) What is the rule for s_n, the n^{th} partial sum?

$$S_n = \sum_{n=0}^{n} x^n$$

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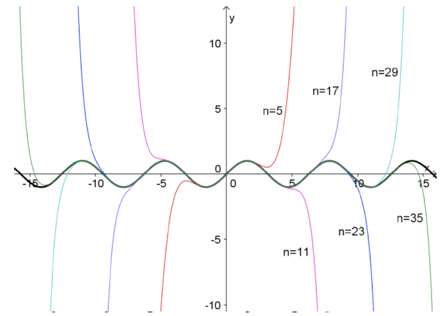
Power Series and the Ratio Test

From Taylor Polynomials to Taylor Series



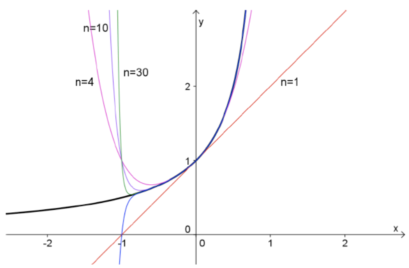
The figure shows, as we have seen, that as we add more terms to the polynomial we get a better fit on a wider interval. In fact, we have seen the pattern that generates these polynomials:

$$f(x) \approx \sum_{k=0}^{n} (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!}.$$

Is it such a stretch to imagine that with infinitely many terms we could make the error go to zero for all x? Can we not imagine that

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = \sin(x) \text{ exactly}$$

As another example, take $g(x) = \dfrac{1}{1-x}$ (Figure 6.2). The polynomials are of the form $P_n(x) = 1 + x + x^2 + \cdots + x^n$. But earlier in the chapter we expanded this as a geometric series. We *already* know, in some sense, that $g(x) = \sum_{n=0}^{\infty} x^n$.



The picture, though, is different from what we saw in Figure 6.1. When we graph a partial sum of the infinite geometric series $1 + x + x^2 + \cdots$ (i.e., a Taylor polynomial), we do not see the fit extend forever. For one thing, no matter how high we push the degree, no polynomial seems to be able to model the unbounded behavior in the graph of g near $x = 1$. The underlying issue here is of convergence. The common ratio of the geometric series $1 + x + x^2 + \cdots$ is x, so this series does not converge if $|x| \geq 1$; the series just doesn't make sense for such x.

POWER SERIES AND CONVERGENCE

Definition A **power series centered at $x = a$** is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n \tag{1}$$

We also allow the initial value of the index to be a positive integer (instead of zero).

Drawing by E. Goldberg

We can think of power series as "Energizer polynomials": they keep on going and going and going...

Thus a power series is an infinite series like those seen in Section 1, but it is also a function of x. Notice that if $a = 0$, the series takes on the relatively simple form $\sum_{n=0}^{\infty} c_n x^n$. The c_ns are just numbers.

$$\sum_{n=0}^{\infty} n x^n = 0 + x + 2x^2 + 3x^3 + \cdots$$

Does a power series converge or diverge?

Power series, however, are more complicated than the series we studied in Section 1 because they depend on x. A particular power series may converge for some values of x but not for others. The set of x-values for which the power series converges is called the interval of convergence.

One nice feature about power series is that the sets of x-values for which they converge are fairly simple. They always converge on a (sometimes trivial) interval, and the center of the series is always right in the middle of that interval. The distance from the center of this interval to either endpoint is called the radius of convergence of the series, often denoted R.

It turns out that there are only three possible cases that can come up:

1. The "bad" case is where $R = 0$. A series with $R = 0$ converges only at its center. An example of such a series is $\sum_{n=0}^{\infty} n!x^n$.
2. The "best" case is where $R = \infty$. By this we mean that the radius of convergence is infinite; the series converges for *all* x-values. As we will show, this is the case for the series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ which represents the sine function.
3. The other case is where $R = L$, some positive, finite number. In this case, the series converges for $|x-a| < R$ and diverges for $|x-a| > R$. In other words, the series converges for x-values within R units of the center of the series but diverges for x-values that are more than R units from the center. (An alternate symbolic expression of this idea is to say that the series converges if $a - R < x < a + R$ and diverges if either $x < a - R$ or $x > a + R$.) What happens if $|x-a| = R$ depends on the particular series.

Definition A **power series centered at $x = a$** is an infinite series of the form $$\sum_{n=0}^{\infty} c_n (x-a)^n \qquad (1)$$ We also allow the initial value of the index to be a positive integer (instead of zero).	

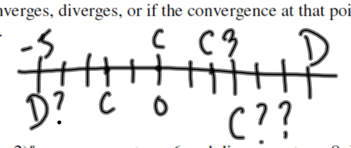
Some series involving x that are NOT power series!

$$\sum (\sin x)^n \qquad \sum \frac{1}{x^n} \qquad \sum \left(x^2-5\right)^n$$

IMPORTANT: The rules about convergence DO NOT work for these series.

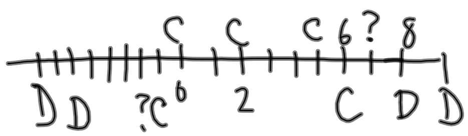
Example 1

The power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = 3$ and diverges for $x = -5$. For each of the following x-values, state whether the series converges, diverges, or if the convergence at that point cannot be determined: -6, −4, -3, -2, 0, 2, 4, 5, and 6.



Practice 1

The power series $\sum_{n=0}^{\infty} c_n (x-2)^n$ converges at $x = 6$ and diverges at $x = 8$. For each of the following x-values, state whether the series converges, diverges, or if the convergence at that point cannot be determined: -8, -6, -2, -1, 0, 2, 5, 7, and 9.



We saw in Section 2 that term-by-term calculus operations produced new Taylor polynomials from old ones. This is true of power series as well. We can differentiate or integrate a power series, one term at a time. This is a very convenient way to create new power series or analyze existing ones. One fact that makes this kind of manipulation simple is that term-by-term operations do not change the radius of convergence of a series.

Example 2

Find the radius of convergence of the power series $\boxed{f(x)=\sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^n}$ and $g(x)=\sum_{n=1}^{\infty}\frac{n}{2^n}(x-3)^{n-1}$.

$$f(x) = 1 + \left(\frac{x-3}{2}\right) + \left(\frac{x-3}{2}\right)^2 + \dots$$

Series is geometric, converges if $|\text{ratio}| < 1$

$\left|\frac{x-3}{2}\right| < 1$ when $|x-3| < 2$ radius is 2, center is 3

THE RATIO TEST

In Example 2 we found the radius of convergence of a geometric power series by using the fact that $|r|$ must be less than 1 for a geometric series to converge. If a series is not geometric, and we cannot easily relate it to one by differentiation or integration, we are out of luck.

The idea behind the ratio test is to see if, long term, the given series behaves like a geometric series. For a geometric series, $\frac{a_{n+1}}{a_n}$ is constant—the common ratio of the series—and only if $\left|\frac{a_{n+1}}{a_n}\right| < 1$ do the terms shrink fast enough for the series to converge. In the ratio test, we no longer suppose that the ratio of successive terms is constant, but we are saying that if that ratio eventually gets—and stays—below 1, then the terms will decrease at a rate on par with a convergent geometric series.

Theorem 6.1 – The Ratio Test
Let $\sum a_n$ be a series in which $a_n > 0$ for all n (or at least all n past some particular threshold value N).

Form the ratio $\dfrac{a_{n+1}}{a_n}$ and evaluate its limit as $n \to \infty$. Provided this limit exists, there are three possible cases.

If $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$, then $\sum a_n$ diverges. (As a bonus, we include $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$ in this case.)

If $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$, then $\sum a_n$ converges.

If $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1$, then the ratio test is inconclusive. $\sum a_n$ could either converge or diverge; another tool is needed to decide the series.

Example 3
Use the ratio test to determine whether the following series converge.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1 \text{ converges}$$

$$(n+1)! = (n+1)n!$$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n}$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{n+1} \cdot \frac{n}{3^n} = 3 \cdot \frac{n}{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} 3 \cdot \frac{n}{n+1} = 3 > 1 \text{ divergence}$$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\sqrt{2n+2}}}{\frac{1}{\sqrt{2n}}} = \frac{\sqrt{2n}}{\sqrt{2n+2}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\sqrt{2n}}{\sqrt{2n+2}} = 1 \text{ No answer!}$$