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$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

$$\frac{\cancel{2n+1}}{\cancel{(2n+1)}(2n)!}$$

$$\cos x \approx 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} \cdots + (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!}$$

$$\approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\frac{\sin x}{x} \approx \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x \to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots$$

$$= 1$$

Can we find a Taylor polynomial for cos(x^2)

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!}$$

$\int \cos(x^2)\,dx$? not easy

I would now like to change gears and come up with a Maclaurin polynomial for $f(x) = \frac{1}{1-x}$.

My preferred approach is to view $\frac{1}{1-x}$ as having the form $\frac{a}{1-r}$ where $a = 1$ and $r = x$. That means, that $\frac{1}{1-x}$ represents the sum of a geometric series with initial term 1 and common ratio x.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

Conv. if $|x| < 1$

As you look at the graphs, a few things should strike you. They are nothing new. All the polynomials appear to match the parent function $f(x)$ perfectly at the center: $x = 0$. As you move away from the center, the quality of the approximation decreases, as seen by the green graphs of the polynomials splitting away from the black graph of f. Finally, adding more terms seems to improve the quality of the approximation; the higher-order polynomials "hug" the graph of f more tightly. The big three features continue to hold. However, we do *not* appear to be able to extend the interval indefinitely as we did with the earlier examples.

Now we must turn to the question of why a Taylor polynomial for $f(x)=\frac{1}{1-x}$ is interesting. Well, it isn't. But we can integrate it, and that *will* be interesting. If we know, for example, that

$$\frac{1}{1-t}\approx 1+t+t^2+t^3,$$

then it should follow that

$$\int_0^x \frac{1}{1-t}dt \approx \int_0^x \left(1+t+t^2+t^3\right)dt.$$

t is just a dummy variable here. Carrying out the integration gives the following.

$$-\ln(1-t)\Big|_0^x \approx \left(t+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{4}\right)\Bigg|_0^x$$

$$-\ln(1-x)\approx x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\tfrac{1}{4}x^4$$

$$\ln(1-x)\approx -x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4$$

From here he we generalize.

$$\ln(1-x)\approx -x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\cdots-\tfrac{1}{n}x^n \qquad (2.4)$$

A Systematic Approach to Taylor Polynomials

How do we find Taylor polynomials that match particular functions?

Generate a Maclaurin Polynomial for: e^x

$f(x)=c_0+c_1x+c_2x^2+c_3x^3$ $\quad f(0)=1$

$f'(x)= \quad c_1 + 2c_2x+3c_3x^2$ $\quad f'(0)=1$

$f''(x)= \quad 2\cdot 1c_2+3\cdot 2c_3x$ $\quad f''(0)=1$

$f'''(x)= \quad 3\cdot 2\cdot 1c_3$ $\quad f'''(0)=1$

The cubic function and its 1st 3 derivs, should match e^x exactly at $x=0$.

Let's solve for each coefficient in terms of the corresponding derivative so we can "see" the pattern that develops...

$c_0 = 1$

$c_1 = 1$

$c_2 = \frac{1}{2}$

$c_3 = \frac{1}{6}$

The 3rd degree Taylor polynomial for e^x at $x=0$ is

$$P_3 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

infinite taylor series is exactly equal to the function, whereas the taylor polynomial is only an approximation (that has increasing accuracy with increasing degree)

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

center

Definition If $f(x)$ is differentiable n times at $x=a$, then its **n^{th}-degree Taylor polynomial centered at a** is given by $P_n(x)=\frac{f(a)}{0!}+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$ or $P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$. If $a=0$, then we may call the polynomial a Maclaurin polynomial.

Practice 3 fourth degree Taylor poly for f

Find a ~~quartic polynomial~~ f that has the following properties: $f(-1)=3$, $f'(-1)=0$, $f''(-1)=-1$, $f'''(-1)=8$, and $f^{(4)}(-1)=10$.

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$a=-1$

Bonus: Does f have a relative extremum at $x = -1$? If so, what kind?

Yes, max by 2nd deriv test.

$$P_4(x) = \frac{3}{0!} + \frac{0}{1!}(x-(-1))^1 + \frac{-1}{2!}(x-(-1))^2 + \frac{8}{3!}(x-(-1))^3 + \frac{10}{4!}(x-(-1))^4$$

Observation: To uniquely determine an n^{th}-degree polynomial, we need $n+1$ *pieces of information* about the function.

"Like...isn't this stuff just so cool! Everything from calculus, everything comes together with this stuff! So cool!"

Try page 43 #10

Writing Taylor Polynomials February 28, 2017

2^{nd} order = go out to $\frac{f''(a)}{2!}$ term

$P_2(x) = 3 - 8x + \frac{5x^2}{2}$ $\cancel{P_3(x)}$ $P_2(0.3) = 0.825$

$P_3(x) = P_2(x) + \frac{2x^3}{6}$ $P_3(0.3) = 0.834$

#1b) $f(x)=e^x$ $a=e$ $n=4$

$f(e)=e^e$

$f'(e)=e^e$

$P_4(x) = e^e + \frac{e^e}{1!}(x-e) + \dots +$