HW: Page 525 #36, 40, 43

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$\int_0^1 \frac{1}{x^p}dx$

DO NOW:

$\int_1^\infty \frac{1}{x^p}dx$

Given:
1) Show this converges for p>1
2) Determine its value for p>1
3) Show that this diverges for p<1
4) What happens when p=1?

$\boxed{\int_1^\infty \frac{1}{x}dx \text{ diverges}}$

Remember

$\frac{1}{x^p} = x^{-p}$

$\int x^{-p}dx = \frac{x^{-p+1}}{-p+1}$ $p \neq 1$

$\lim_{x\to\infty} \frac{1}{x^p} = 0$ if $p>0$

$\lim_{x\to\infty} x^{-p} = 0$ if $p>0$

$$\int_1^\infty x^{-p}dx = \lim_{t\to\infty}\int_1^t x^{-p}dx$$

$$= \lim_{t\to\infty} \left.\frac{x^{1-p}}{1-p}\right]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}\right)$$

If $1-p<0$ then $\lim_{t\to\infty} t^{1-p} = 0$ or $p>1$

for $p>1$ $= \frac{-1}{1-p}$

$= \frac{1}{p-1}$

for $p<1$, divergence

If $1-p>0$, $\lim_{t\to\infty} t^{1-p} = \infty$

Need to Know

$\int_1^\infty \frac{1}{x^p}dx$ conv if $p>1$

div if $p\leq 1$

When and how do we "split" integrals?

(7) $\int_0^\infty \frac{dx}{x^2}$

$\int_1^\infty \frac{dx}{x^2} = 1$ $\quad \int_0^1 \frac{dx}{x^2}$ diverges

(33) $\int_1^\infty \frac{dx}{x(\ln x)^2} = \int_1^2 + \int_2^\infty$

Has to split

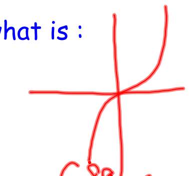
$$\int_1^\infty \frac{1}{(x-2)^3(x-3)^2}\,dx$$

$$\int \frac{\arctan x}{1+x^2}\,dx = \frac{(\arctan x)^2}{2} + C$$

$u = \arctan x \quad du = \frac{1}{1+x^2}dx$

Is INFINITY a number?

For any finite number a, what is : $\int_{-a}^{a} x^3\,dx = 0$



So, what is: $\int_{-\infty}^{\infty} x^3\,dx = \underbrace{\int_{-\infty}^{88} x^3\,dx}_{\text{div}} + \underbrace{\int_{88}^{\infty} x^3\,dx}_{\text{div}}$

Another Question:

What if we can't find an antiderivative, or it's hard to find, or it's hard to take the limit as the upper limit goes to infinity?

Example: $\int_1^\infty \frac{1}{x^5+1}dx$ We'll ask Wolfram Alpha for help with this one

Indefinite integral: Approximate form Step-by-step solution

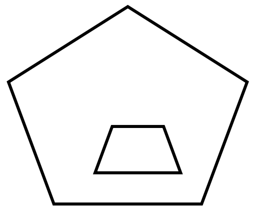
$$\int \frac{1}{1+x^5}\,dx = \frac{1}{20}\left(-(1+\sqrt{5})\log(2x^2+(-1-\sqrt{5})x+2)+(\sqrt{5}-1)\log(2x^2+(\sqrt{5}-1)x+2)+4\log(x+1)-2\sqrt{10-2\sqrt{5}}\tan^{-1}\left(\frac{-4x+\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}\right)+2\sqrt{2(5+\sqrt{5})}\tan^{-1}\left(\frac{4x+\sqrt{5}-1}{\sqrt{2(5+\sqrt{5})}}\right)\right) + \text{constant}$$

$\tan^{-1}(x)$ is the inverse tangent function
$\log(x)$ is the natural logarithm

An Easier Solution: Think back to the idea of area: If "shape1" contains "shape2" then area(shape1)>area(shape2)

On the interval $1 \le x < \infty$, $\dfrac{1}{x^5} > \dfrac{1}{x^5+1}$

so if $I_1 = \int_1^\infty \frac{1}{x^5}dx > \int_1^\infty \frac{1}{x^5+1}dx = I_2$



This doesn't give us a precise value for I_2 but we know it is some particular value less than $I_1 = 1/4$

Comparison Test for Nonnegative Improper Integrals

Let f and g be continuous functions. Suppose for all $x \ge a$, $0 \le f(x) \le g(x)$

1) If $\int_a^\infty g(x)dx$ converges, then so does $\int_a^\infty f(x)dx$

2) If $\int_a^\infty f(x)dx$ diverges, then so does $\int_a^\infty g(x)dx$

In order to use this test, we need to guess whether an integral converges or diverges and then find an appropriate function to compare to.

Some functions we can use for comparison:

$\int_0^\infty e^{-x}dx$

$\int_1^\infty \frac{1}{x^p}dx$ for $p>1$ $\boxed{\text{Converge}}$

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EXAMPLES:

Example 1: We might need to do a little work, for instance:

$\int_0^\infty e^{-x^2}dx$ can be compared to xe^{-x^2} on $1 \le x < \infty$

$xe^{-x^2} \ge e^{-x^2}$

Since we don't care what happens on intervals where the integral is not improper.

$\int_0^1 e^{-x^2}dx$ not imprope

Example 2: $\int_1^\infty \frac{1}{x+1}dx$ can be compared to $\int_1^\infty \frac{1}{2x}dx$

NB: We will not worry about "how close" we can get to the actual values of improper integrals. [This topic is covered in the textbook.]

$\frac{1}{x+1} < \frac{1}{x}$

$\frac{1}{x+1} > \frac{1}{2x}$

http://www.math.uri.edu/~pakula/142sec3f11/limcomp.pdf

The Limit Comparison Test for Improper Integrals

The following test is often, but not always, a useful alternative to the comparison test given on p. 381 of the textbook.

Limit Comparison Test. If $f(x)$ and $g(x)$ are both positive when $x \geq a$ and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \text{ and } 0 < L < \infty$$

then the improper integrals

$$\int_a^\infty f(x)\,dx \text{ and } \int_a^\infty g(x)\,dx$$

are either *both convergent* or *both divergent.*

Example. Is $\int_2^\infty \frac{1}{\sqrt{x+1}}\,dx$ convergent? The obvious integral to use for comparison is $\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ which we know diverges because $\int_2^\infty \frac{1}{x^p}\,dx$ diverges when $p < 1$. However,

$$\frac{1}{\sqrt{x+1}} \leq \frac{1}{\sqrt{x}}$$

so the obvious comparison is not the one we want. (We would want the opposite inequality!) On the other hand, using the Limit Comparison Test we find

$$\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x+1}}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \sqrt{\frac{x}{x+1}} = \sqrt{1} = 1.$$

Here $L = 1$ and $0 < 1 < \infty$ so we conclude that our integral is divergent.

1

$$\int_1^\infty xe^{-x^2}dx = -\frac{1}{2}\int_{-1}^{-\infty} e^u\,du$$

$u = -x^2 \quad du = -2x\,dx$

$x\,dx = -\frac{1}{2}du$

$$= -\frac{1}{2}\lim_{t\to-\infty}\int_{-1}^{t} e^u\,du$$

$$= -\frac{1}{2}\lim_{t\to-\infty} e^u\Big]_{-1}^{t} = \frac{1}{2e}$$

15 $\int_0^\infty \frac{dx}{x^4+1} < \int_0^\infty$

$\int_0^1 \frac{1}{x^4+1}\,dx$

$\frac{1}{x^4+1} < \frac{1}{x^4}$

$\int_1^\infty \frac{dx}{x^4+1} < \int_1^\infty \frac{1}{x^4}\,dx = \frac{1}{3}$