GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 1998

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. When 72% of $\left(\frac{3}{4} + \frac{1}{3}\left(\frac{4}{3}\right)^{-2}\right)$ is put the form $\frac{a}{b}$ where a and b are relatively prime whole numbers, find $a + b$.

2. Find the **sum** of the three smallest non-prime two digit whole numbers each of whose digits is prime.

3. How many natural numbers less than 100 are **not** divisible by 2 or 3?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 1998

ROUND 2 – Algebra 1 – Word Problems

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The ratio of three natural numbers is 3:5:12. Three times the smallest number increased by twice the largest number is twenty-one more than six times the remaining number. Find the sum of these three natural numbers.

2. Manuel's father's age is now two years more than four times Manuel's age. In six years Manuel will be half of what his father's age was when Manuel was born. How old is Manuel now?

3. Three trains leave from the same location thirty minutes apart going along the exact same straight track. The first train to leave is travelling at a speed eight kilometers per hour slower than the second train. The third train's speed is two kilometers per hour faster than the first train. Three hours after the first train has left, all trains are still travelling along the track. If the distance then between the second and third trains is three times the distance between the first and second trains, what is the speed of the first train in kilometers per hour?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 1998

ROUND 3 – Algebra 1 – Exponents and Radicals

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Write in simplest radical form: $5\left(20^{-1/2}\right) + \left(3 + \sqrt{5}\right)^{-1} - \left(1\,{}^{7}\!/_{9}\right)^{-1/2}$

2. Given $\dfrac{\sqrt{6}}{\sqrt{3} - \sqrt{2}} + \dfrac{\sqrt{12}}{\sqrt{6} + \sqrt{2}} = a\sqrt{2} + b\sqrt{3} + c\sqrt{6}$, where a, b, and c are rational, find the product abc.

3. Solve for x: $2^{x+1} + 2^{x+2} = 4^{19} - 4^{18}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 1998

ROUND 4 – Algebra 2– Factoring

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Factor the following into the product of two polynomials: $a^2 + b^2 - c^2 - d^2 - 2ab - 2cd$

2. Factor the following into the product of two polynomials: $x^3 - 8y^3 + 3x^2 + 3x + 1$

3. Factor the following: $4^x - x^4 + 4^2 - 2^{x+3}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 1998

ROUND 5 – Trigonometry: Angular and Linear Velocity; Right Triangle

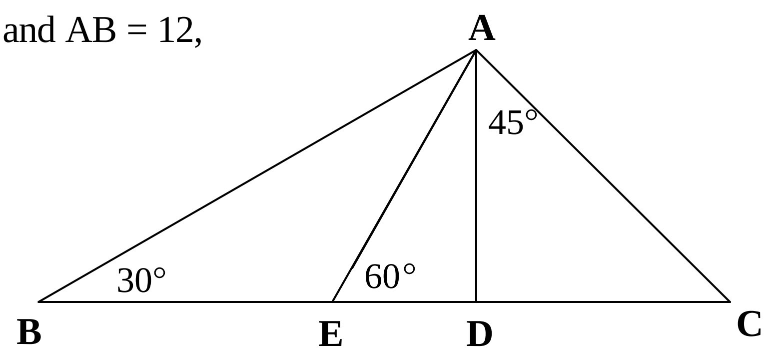
1. ____________________

2. ____________________

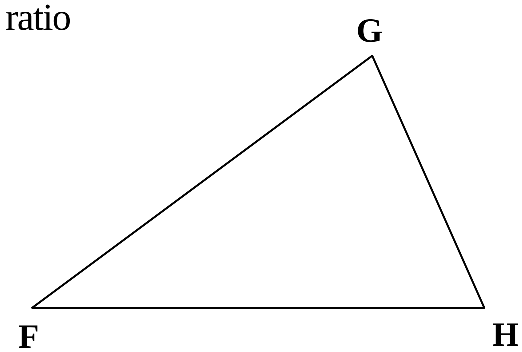
3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE

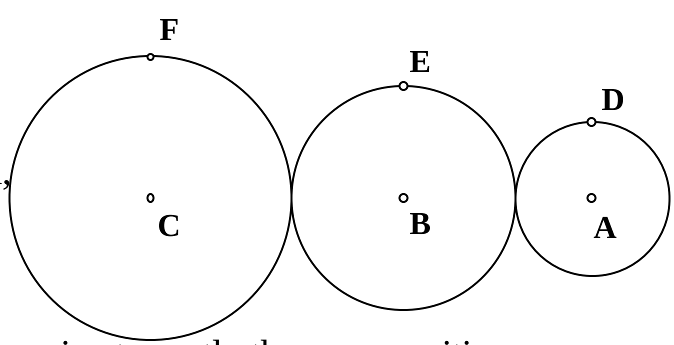
1. Given the diagram with $\overline{AD} \perp \overline{BC}$ and AB = 12, find AC – BE.



2. Given $\tan \angle F = \frac{3}{4}$ and $\tan \angle H = \frac{24}{7}$, find the ratio of GF to FH.



3. Given externally tangent circles centered at A, B, and C with radii of 8cm, 12cm, and 15cm respectively, circle A is rotating about A at $\frac{\pi}{5}$ radians per second, which rotates circle B about B, which in turn rotates circle C about C. Find the fewest number of seconds so that points D, E, and F will be located again at exactly the same positions as now.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 1998

TEAM ROUND

3 pts. 1. ___________________

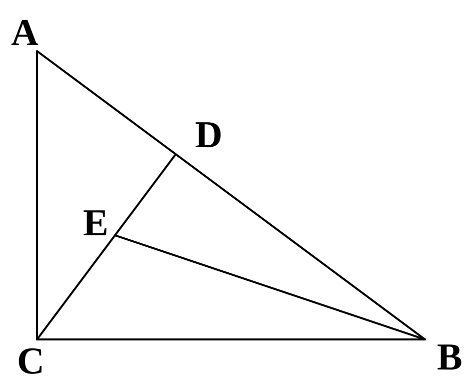
3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND

1. Given that the division, $2x5y_6 \div 15_6$, in base 6 produces a whole number result, find all possible ordered pairs (x,y).

2. Given AC = 3, BC = 4, $\angle$ ACB is right, $\angle$ ADC is right, and $\overline{BE}$ bisects $\angle$ ABC, find the distance from E to $\overline{BC}$.



3. Find the last two digits in $2^{1998} + 3^{1998}$.

Detailed Solutions of GBML MEET 1 – OCTOBER 1998

ROUND 1

1. 72% of $\left(\frac{3}{4}+\frac{1}{3}\left(\frac{4}{3}\right)^{-2}\right)=\frac{18}{25}\left(\frac{3}{4}+\frac{3}{16}\right)=\frac{18}{25}\cdot\frac{15}{16}=\frac{27}{40}\Rightarrow a+b=\mathbf{67}$

2. The three smallest two digit non-prime whole numbers each of whose digits is prime are 22, 25, and 27. Their sum equals **74**

3. Method 1: There are 99 numbers altogether; $2\cdot1, 2\cdot2, \ldots 2\cdot49$ are divisible by 2; $3\cdot1, 3\cdot2, \ldots 3\cdot33$ are divisible by 3; $6\cdot1, 6\cdot2, \ldots 6\cdot16$ are divisible by 6, which are numbers in the multiples of 2 list and the multiples of 3 list $\Rightarrow 99 - 49 - 33 + 16 = 33$
Method 2: In mod 6, 1,5 are not divisible by 2 or 3. The 99 numbers are 16 groups of 6 and 1 group of 3 $\Rightarrow 16\cdot2 + 1 = 33$ numbers not divisible by 2 or 3.

ROUND 2

1. Call the three numbers $3x$, $5x$, and $12x$; Equation: $9x + 24x = 30x + 21 \Rightarrow x = 7$;
Sum of the numbers = $20x = 140$

2. Manuel's age = x ; Manuel's father's age = $4x + 2$
Manuel's age in 6 years = $x + 6$; Manuel's father's age when Manuel was born = $3x + 2$
Equation: $2(x+6) = 3x + 2 \Rightarrow x = 10$

3.

	rate	time	distance
1st train	$x - 8$	3	$3x - 24$
2nd train	x	2.5	$2.5x$
3rd train	$x - 6$	2	$2x - 12$

Equation:
$2.5x - 2x + 12 = 3(3x - 24 - 2.5x)$
$\Rightarrow x = 84 \Rightarrow x - 8 = 76$

ROUND 3

1. $5\left(20^{-½}\right) + \left(3 + \sqrt{5}\right)^{-1} - \left(1\tfrac{7}{9}\right)^{-½} = 5\left(\dfrac{1}{\sqrt{20}}\right) + \dfrac{1}{3 + \sqrt{5}} - \left(\dfrac{9}{16}\right)^{½} = \dfrac{\sqrt{5}}{2} + \dfrac{3 - \sqrt{5}}{4} - \dfrac{3}{4} = \dfrac{\sqrt{5}}{4}$

2. $\dfrac{\sqrt{6}}{\sqrt{3} - \sqrt{2}} + \dfrac{\sqrt{12}}{\sqrt{6} + \sqrt{2}} = \sqrt{6}\left(\sqrt{3} + \sqrt{2}\right) + \dfrac{2\sqrt{3}\left(\sqrt{6} - \sqrt{2}\right)}{4} =$
$3\sqrt{2} + 2\sqrt{3} + \dfrac{3}{2}\sqrt{2} - \dfrac{1}{2}\sqrt{6} = \dfrac{9}{2}\sqrt{2} + 2\sqrt{3} - \dfrac{1}{2}\sqrt{6} \Rightarrow abc = -\dfrac{9}{2}$ or - 4.5

3. $2^{x+1} + 2^{x+2} = 4^{19} - 4^{18} \Rightarrow 2^{x+1}\left(1 + 2\right) = 4^{18}\left(4 - 1\right) \Rightarrow 2^{x+1} = 4^{18} \Rightarrow 2^{x+1} = 2^{36} \Rightarrow x = 35$

ROUND 4

1. $a^2 + b^2 - c^2 - d^2 - 2ab - 2cd = \left(a^2 - 2ab + b^2\right) - \left(c^2 + 2cd + d^2\right) =$
$\left(a - b\right)^2 - \left(c + d\right)^2 = \left(a - b - c - d\right)\left(a - b + c + d\right)$

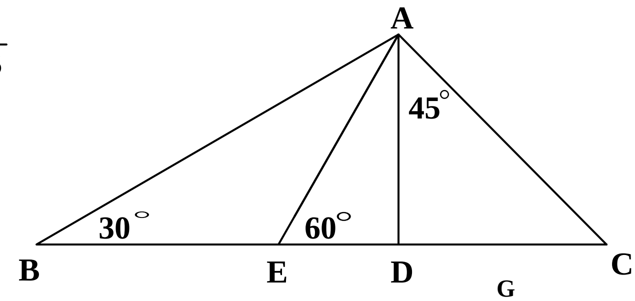
2. $x^3 - 8y^3 + 3x^2 + 3x + 1 = x^3 + 3x^2 + 3x + 1 - 8y^3 = \left(x + 1\right)^3 - \left(2y\right)^3 =$

$\left(x + 1 - 2y\right)\left(\left(x + 1\right)^2 + 2y\left(x + 1\right) + \left(2y\right)^2\right) = \boldsymbol{\left(x + 1 - 2y\right)\left(x^2 + 2x + 1 + 2xy + 2y + 4y^2\right)}$

3. $4^x - x^4 + 4^2 - 2^{x+3} = 2^{2x} - 8 \cdot 2^x + 4^2 - \left(x^2\right)^2 = \left(2^x - 4\right)^2 - \left(x^2\right)^2 = \boldsymbol{\left(2^x - x^2 - 4\right)\left(2^x + x^2 - 4\right)}$

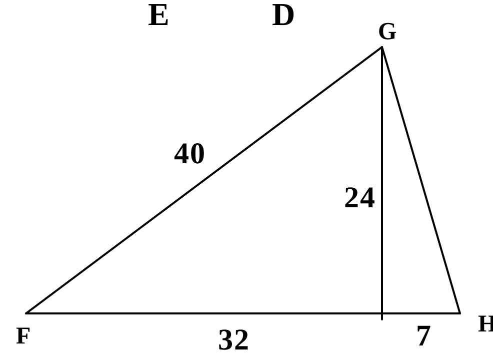
ROUND 5

1. $AB = 12 \Rightarrow AD = 6 \Rightarrow AC = 6\sqrt{2}$, $BD = 6\sqrt{3}$ and $DE = 2\sqrt{3} \Rightarrow BE = 4\sqrt{3} \Rightarrow$ $AC - BE = 6\sqrt{2} - 4\sqrt{3}$



2. $\frac{3}{4} = \frac{24}{32}$ Draw a perpendicular from G to FH

ratio of GF to FH = 40:39

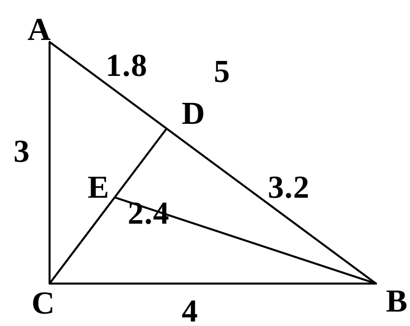


3. For points D, E, and F to be located in the exact same location a second time, each circle needs to rotate some whole number of revolutions. Call the circumferences of circles A, B, and C **C_A, C_B,** and **C_C ; C_A =** 16π cm; **C_B =** 24π cm; **C_C =** 30π cm; The least common multiple of these three circumferences is 240π cm, which is 15 revolutions of circle A. One revolution of circle A takes 10 secs. ⇒ 15 revolutions take 150 sec.

TEAM ROUND

1. $2x5y_6 \div 15_6 = \left(2 \cdot 216 + 36x + 5 \cdot 6 + y\right) \div \left(1 \cdot 6 + 5\right) = \left(462 + 36x + y\right) \div 11$
If this is a whole number and since 462 is divisible by 11, then $36x + y$ is divisible by 11 ⇒ $3x + y$ is divisible by 11; x and y are integers from 0 to 5; If $x = 0 \Rightarrow y = 0$. If $x = 1$, $y = 8$, impossible. If $x = 2$, $y = 5$. If $x = 3$, $y = 2$. If $x = 4$, $y = 10$, impossible. If $x = 5$, $y = 7$, impossible. ⇒ **(0,0) (2,5) and (3,2)** are the solutions.

2. The distance from E to $\overline{BC}$ = DE
The ratio of DE: EC = 4:5 ⇒
$DE = \frac{4}{9}\left(2.4\right) = \frac{4}{9} \cdot \frac{12}{5} = \frac{16}{15} \approx 1.0667$



3. Using a calculator to generate powers of 2 and 3, you observe that 2^{22} ends in 04 and 3^{22} ends in 09 ⇒ The last 2 digits of the powers of 2 and 3 repeat every 20 times; 1998 = 18 mod 20 ; 2^{18} ends in 44 and 3^{18} ends in 89; 44 + 89 = 133 ⇒ **33** is the answer.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 1 – OCTOBER 1998

ANSWER SHEET:

ROUND 1

1. 67
2. 74
3. 33

ROUND 2

1. 140
2. 10
3. 76

ROUND 3

1. $\frac{\sqrt{5}}{4}$
2. $-\frac{9}{2}$ or -4.5
3. 35

ROUND 4

1. $(a-b-c-d)(a-b+c+d)$
2. $(x+1-2y)(x^2+2x+1+2xy+2y+4y^2)$
3. $(2^x-x^2-4)(2^x+x^2-4)$

ROUND 5

1. $6\sqrt{2}-4\sqrt{3}$
2. 40:39
3. 150

TEAM ROUND

3 pts. 1. (0,0), (2,5), (3,2)

3 pts. 2. $\frac{16}{15} \approx 1.0667$

4 pts. 3. 33

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 1999

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the following addition in **base 8**, shown below, compute the base 10 sum, $a + b + c$.

$$\begin{array}{r} a\,5\,7 \\ 1\,6\,c \\ 2\,b\,6 \\ \hline 7\,5\,1 \end{array}$$

2. A stock increased in price 25% after one year and then increased $33\frac{1}{3}$% over that price at the end of the second year. After the third year, it was still 10% more than its original price. Compute the percent decrease of the stock after the third year from its price at the end of the second year.

3. How many natural numbers between 267 and 511 are divisible by 4 or 5?

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GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 1999

ROUND 2 – Algebra 1 – Word Problems

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A motorist and a bicyclist set out in opposite directions from the same location, the motorist leaving at 8:00 AM, the bicyclist at 8:40 AM. The motorist is travelling twice as fast as the bicyclist and at 12:20 PM on that same day they are 296 kilometers apart. Compute the speed of the motorist in kilometers per hour.

2. A 16 ounce can of nuts contain 10 ounces of peanuts costing $.08 per ounce and the rest cashews costing $.40 per ounce. If a certain amount of peanuts are replaced with cashews, and the price of the nuts in the can increases 25%, compute the number of ounces of peanuts that are now in the can.

3. Three positive numbers add to 30 with one twice another. If twice the sum of the two smallest is nine more than the largest, compute the smallest of the three numbers.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 1999

ROUND 3 – Algebra 1 – Exponents and Radicals

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Compute the following:
$$\frac{\left(\sqrt[3]{5}\right)\left(\sqrt[4]{25}\right)}{\left(\sqrt[6]{0.2}\right)}$$

2. $\sqrt{44+16\sqrt{6}}$ in simplest radical form equals $a\sqrt{b}+c\sqrt{d}$, where $a, b, c,$ and d are all positive integers. Compute the product $a\,b\,c\,d$.

3. Compute all real solutions to the equation, $\sqrt{5x-4}-\sqrt{x+8}=2$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 1999

ROUND 4 – Algebra 2– Factoring

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Factor the following completely: $12x^3 - 46x^2 + 42x$

2. Factor the following into the product of 2 polynomials: $4x^3 - 9xy^2 + 10x + 15y$

3. Factor the following into the product of 2 polynomials: $x^2 - a - ax - 3x - 4$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 1999

ROUND 5 – Trigonometry: Angular and Linear Velocity; Right Triangle

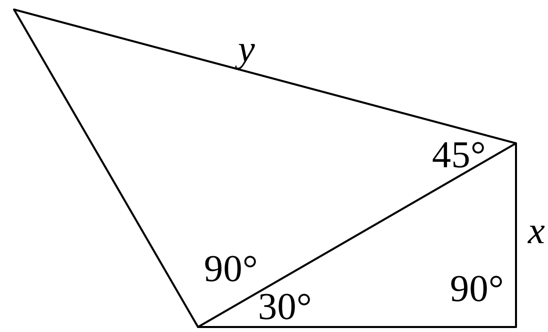
1. ______________________

2. ______________________

3. ______________________

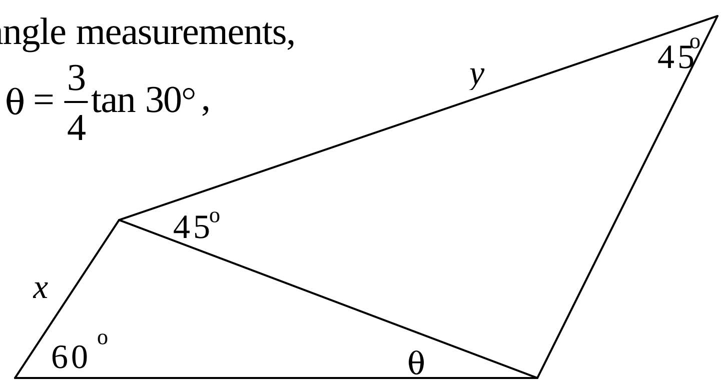
CALCULATORS ARE NOT ALLOWED ON THIS ROUND
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE

1. Given the diagram on the right with indicated lengths x, y and angle measurements. Compute the ratio of x to y.



2. A car is travelling at 25 meters per second and has a wheel radius of 375 millimeters. How many minutes does it take a point at the bottom of the wheel to turn through 800 revolutions?

3. Given the diagram with indicated angle measurements, indicated lengths x and y, and $\tan\theta = \frac{3}{4}\tan 30^\circ$, compute the ratio of y to x.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 1999

TEAM ROUND

5 pts. 1. ___________________

5 pts. 2. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. How many 3-digit whole numbers have exactly 10 positive integral factors?

2. Given θ is an acute angle of a right triangle whose sides have whole number lengths and csc θ = $x + 0.05$, where x is a whole number, compute sec θ. Put the result(s) in the form $\frac{a}{b}$ where a and b are relatively prime whole numbers.

Detailed Solutions for GBML MEET 1 – SEPTEMBER 1999

ROUND 1

1. $\begin{array}{r} a\,5\,7 \\ 1\,6\,c \\ 2\,b\,6 \\ \hline 7\,5\,1 \end{array}$. Since the right column adds to 1, and 17 = 1 mod 8 $\Rightarrow c = 4$. There is a 2 carry to the middle column, and since the middle column adds to 5 = 13 mod 8 $\Rightarrow b = 0$. There is a 1 carry to the left column which adds to 7 $\Rightarrow a = 3 \Rightarrow a + b + c =$ **7**

2. $\frac{5}{4} \cdot \frac{4}{3} x = \frac{11}{10} \Rightarrow x = \frac{33}{50} = 66\% \Rightarrow \mathbf{34\%}$ decrease

3. 267÷4 = 66 R3; 511÷4 = 127 R3; 267÷5 = 53 R2; 511÷5 = 102 R1; 267÷20 = 13 R7; 511÷20 = 25 R11; $\Rightarrow$ there are 127 – 67 + 1 = 61 multiples of 4, there are 102 – 54 + 1 = 49 multiples of 5, and there are 25 – 14 + 1 = 12 multiples of 20 $\Rightarrow$ there are 61 + 49 – 12 = **98** numbers divisible by either 4 or 5.

ROUND 2

1. Let x = speed of the bicyclist and $2x$ = speed of the motorist. Time traveled by motorist = 13/3 hours and time travelled by the bicyclist = 11/3 hours. Equation is:
$\frac{13}{3}2x + \frac{11}{3}x = 296 \Rightarrow \frac{37}{3}x = 296 \Rightarrow \boldsymbol{x} =$ 24 and $2x =$ **48** kph.

2. $10 \times .08 + 6 \times .40 = 3.20$; $3.20 \times 1.25 = 4.00$ (new cost of the nuts)
equation: $.08(10 - x) + .40(6 + x) = 4 \Rightarrow .32x = .80 \Rightarrow x = 2.5 \Rightarrow$ **7.5** oz of peanuts now

3. The three numbers are x, $2x$, and $30 - 3x$
Case I: $2x$ the largest: equation is $2(x + 30 - 3x) = 2x + 9 \Rightarrow 60 - 4x = 2x + 9 \Rightarrow x = \frac{17}{2}$,
and $30 - 3 \cdot \frac{17}{2} = \mathbf{\frac{9}{2}}$
Case II: $30 - 3x$ the largest: equation is $2(x + 2x) = 30 - 3x + 9 \Rightarrow 9x = 39 \Rightarrow x = \mathbf{\frac{13}{3}}$

ROUND 3

1. $$\frac{\left(\sqrt[3]{5}\right)\left(\sqrt[4]{25}\right)}{\left(\sqrt[6]{0.2}\right)} = \frac{\left(5^{1/3}\right)\left(25^{1/4}\right)}{\left(\frac{1}{5}\right)^{1/6}} = \frac{5^{1/3}\cdot 5^{1/2}}{5^{-1/6}} = 5^{1/3+1/2+1/6} = \mathbf{5}$$

2. $\sqrt{44+16\sqrt{6}} = \sqrt{4\left(11+4\sqrt{6}\right)} = 2\cdot\sqrt{11+4\sqrt{6}}$;
$\left(a'\sqrt{b'}+c'\sqrt{d'}\right)^2 = 11+4\sqrt{6} \Rightarrow a'^2b'+c'^2d' = 11$ and $a'c'\sqrt{b'd'} = 2\sqrt{6} \Rightarrow$
$b' = 2,\ d' = 3,\ a' = 2,\ c' = 1 \Rightarrow \sqrt{44+16\sqrt{6}} = 4\sqrt{2}+2\sqrt{3} \Rightarrow a\cdot b\cdot c\cdot d = \mathbf{48}$

3. $\sqrt{5x-4}-\sqrt{x+8}=2 \Rightarrow \sqrt{5x-4}=2+\sqrt{x+8} \Rightarrow 5x-4=x+12+4\sqrt{x+8} \Rightarrow$
$4x-16=4\sqrt{x+8} \Rightarrow x-4=\sqrt{x+8} \Rightarrow x^2-8x+16=x+8 \Rightarrow$
$x^2-9x+8=0 \Rightarrow (x-1)(x-8)=0 \Rightarrow$ Since $x = 1$ is extraneous, $x =$ **8.**

ROUND 4

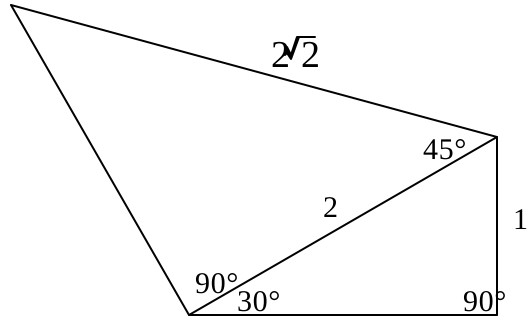
1. $12x^3-46x^2+42x = 2x\left(6x^2-23x+21\right) = \mathbf{2x(3x-7)(2x-3)}$

2. $4x^3-9xy^2+10x+15y = x\left(4x^2-9y^2\right)+5\left(2x+3y\right) = x\left(2x+3y\right)\left(2x-3y\right)+5\left(2x+3y\right) =$
$\left(2x+3y\right)\left(2x^2-3xy+5\right)$

3. $x^2-a-ax-3x-4 = x^2-3x-4-ax-a = \left(x-4\right)\left(x+1\right)-a\left(x+1\right) = \mathbf{(x+1)(x-4-a)}$

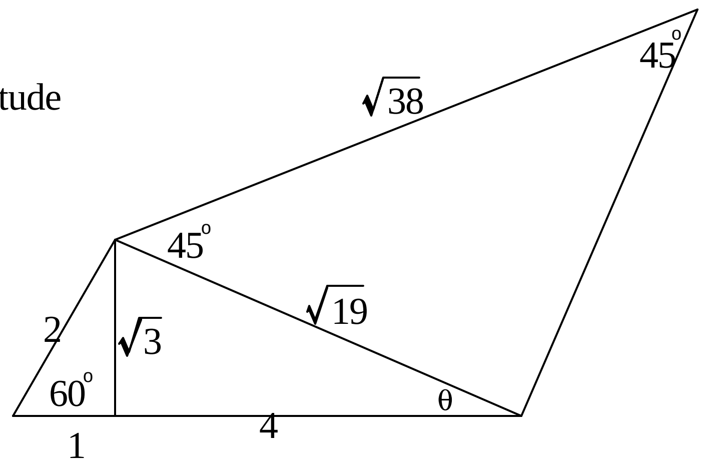
ROUND 5

1. Let $x = 1 \Rightarrow\ y = 2\sqrt{2} \Rightarrow x{:}y = \mathbf{\sqrt{2}:4}$



2. Circumference = 0.75π meters; 0.75π meters × 800 = 600π meters;
600π meters ÷ 1500 meters per minute = **0.4p** minutes

3. Let $x = 2$ for convenience. Draw the altitude as indicated on the diagram.
$\tan\theta = \frac{3}{4}\cdot\frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{4} \Rightarrow$ leg of the 45-45-90° $\Delta = \sqrt{19} \Rightarrow y = \sqrt{38} \Rightarrow$ ratio of y to x = $\mathbf{\sqrt{38}:2}$



TEAM ROUND

1. Any whole number with 10 factors is of the form p^9 or p^4q where p and q are primes
$2^9 = 512$, $3^9 > 1000$ and does not qualify; $2^4 = 16$, q = 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61; $3^4 = 81$, q = 2, 5,7, 11; $5^4 = 625$ and so does not qualify.
Therefore there are **20** possibilities

2. .05 = 1/20 ⇒ one leg has length 20 ⇒ $c^2 - a^2 = 20^2 \Rightarrow (c+a)(c-a) = 20^2$; Now consider all possible sets of simultaneous equations where $c + a$ and $c - a$ are even factor of 20^2
$c + a = 200$, $c - a = 2 \Rightarrow c = 101$ and $a = 99$, which is the only one where c ends in a 1 ⇒
$\sec\theta = \mathbf{\frac{101}{99}}$ [Note c is of the form $20x + 1 \Rightarrow c$ has a units digit equal to 1.]

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 1999

ANSWER SHEET:

ROUND 1

1. 7
2. 34%
3. 98

ROUND 4

1. $2x(3x-7)(2x-3)$
2. $(2x+3y)(2x^2-3xy+5)$
3. $(x+1)(x-4-a)$

ROUND 2

1. 48 (48 kph)
2. 7.5
3. $\frac{13}{3}$, $\mathbf{\frac{9}{2}}$

ROUND 5

1. $\sqrt{2}:4$
2. 0.4π $\left(\frac{2\pi}{5}\right)$
3. $\sqrt{38}:2$

ROUND 3

1. 5
2. 48
3. 8

TEAM ROUND

5 pts. 1. 20

5 pts. 2. $\frac{101}{99}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 2000

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Find the smallest 4-digit whole number divisible by 12, 15, 18, and 24.

2. A stock underwent the following changes over a three month period: 1st month: 20% drop, 2nd month: up 20 points, 3rd month: 30% gain. If the final value of the stock was 78 points, how many points was the stock worth at the beginning of this three month period?

3. The 3-digit base 6 number, xyz_6 , $x > y > z$, is divisible by 2, 3, and 5. Find all possible base 6 numbers xyz_6 satisfying these conditions. Note your answer(s) should be left in base 6.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 2000

ROUND 2 – Algebra 1 – Word Problems

1. ____________________

2. ____________________

3. _____________%___

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. A car and a bicycle set out in the same direction, from the same location on the same morning, the bicyclist leaving at 9:00 AM, the motorist at 10:24 AM. The car is averaging 12 MPH faster than the bicycle and at 12:30 PM on that same day the car has overtaken the bicycle. How many miles has each of the vehicles traveled when they meet?

2. An inheritance was divided among three heirs in the ratio of 4:3:2. If the recipient with the largest share gave $1000 to each of the other heirs, then her amount would be $24,000 less than twice the sum of the other two. What is the total dollar value in the inheritance?

3. Brand A sells cans of mixed nuts advertising 28% cashews, 16% walnuts, and the rest peanuts. Brand B sells its cans of mixed nuts advertising 25% more cashews and 12.5% more walnuts than brand A, and the rest peanuts. If the ratio of the costs per pound of cashews to walnuts to peanuts is 4:2:1 and a can of brand B holds the same weight in nuts as a can of brand A, what percent higher in price should a can of brand B's mixed nuts cost than a can of brand A's mixed nuts?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 2000

ROUND 3 – Algebra 1 – Exponents and Radicals

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Find the value of the expression below in the form $\frac{a}{b}$ where a and b are relatively prime positive integers.

$$\sqrt{7\,{}^{1}\!/_{9}}-\sqrt{\frac{1}{9}+\frac{1}{16}}+\left(3+3^{-1}\right)^{2}$$

2. Solve the following equation for *x*, where $x>0$:

$$\frac{\sqrt[3]{\sqrt[4]{x^{6}}}}{\sqrt{\sqrt[6]{x^{4}}}}=\sqrt{3}\cdot\sqrt[3]{2}$$

3. Simplify the following expression:

$$\left(\frac{2^{-1}}{\sqrt[3]{2}-1}\right)\left(1-2^{1/6}\right)\left(1+2^{1/6}\right)-2^{-2}$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 2000

ROUND 4 – Algebra 2– Factoring

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Factor the following completely: $x^5 - 9x^3 - 8x^2 + 72$

2. Factor the following into the product of 2 polynomials: $2a^2 + 2b^2 - 5a + 5b - 4ab - 12$

3. Factor the following into the product of 2 polynomials: $x^2 - 3a^2 - xy - ay - 2ax$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 2000

ROUND 5 – Trigonometry: Angular and Linear Velocity; Right Triangle

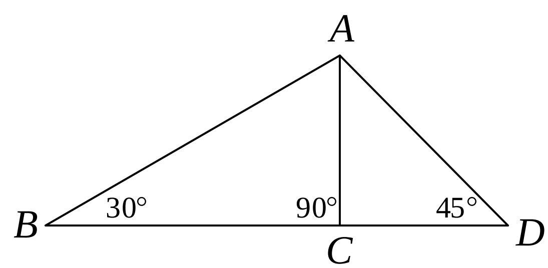
1. ______________________

2. ______________________

3. ______________________

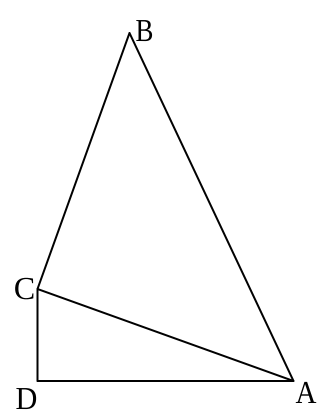
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. Given the diagram on the right, if $\overline{BCD}$ and $BC = 6$, find the length of $\overline{AD}$.



2. The front wheel of an old fashioned bicycle has a radius of 6 inches while the back wheel has a radius of $1\frac{3}{4}$ feet. If the front wheel, while traveling, is rotating at 315 revolutions per minute, the back wheel makes how many revolutions in 1 second?

3. Given $m\angle CAB = m\angle ABC = 45°$, $\tan(\angle CAD) = \frac{\sqrt{2}}{4}$, $m\angle D = 90°$, and AB = 12, find the perimeter of quadrilateral ABCD.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 2000

TEAM ROUND

3 pts. 1. ___________________

3 pts. 2. ___________________

4 pts. 3.____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. The greatest common factor of three distinct natural numbers 21, x, and y is 21, and their least common multiple is 462. How many ordered pairs (x, y), $x < y$, satisfy these conditions?

2. Given the polynomial in x and y, $4x^4 + ky^4$, is factorable over the rationals, write the factored form for this polynomial for the minimum value for k, where k is an odd integer greater than 1.

3. Right triangle ABC has legs $\overline{AC}$ and $\overline{BC}$. If AC = 105 and the lengths of $\overline{AB}$ and $\overline{BC}$ are whole numbers, find the smallest possible value for $\sin(\angle A)$ as a rational number in reduced form.

Detailed Solutions for GBML MEET 1 – SEPTEMBER 2000

ROUND 1

1. The LCM of 12, 15, 18, and 24 = $8 \times 9 \times 5$ = 360.
   The smallest 4-digit multiple of 360 = 360×3 = 1080

2. Let x = original value of the stock:
   $1.3(.8x + 20) = 78 \rightarrow .8x + 20 = 60 \rightarrow .8x = 40 \rightarrow x = 50$

3. Since xyz_6 is divisible by 2, 3, and 5, it must be divisible by 6 $\rightarrow z = 0 \rightarrow$ the base 10 value of the number = $36x + 6y = 6(6x + y)$; this is divisible by 30 $\rightarrow$ $6x + y$ is divisible by 5; since $x > y > z = 0$, the only possibilities are $x = 3$, $y = 2$ or $x = 4$, $y = 1 \rightarrow$ answers are 320_6 and 410_6.

ROUND 2

1. Let x = speed of the bicycle $\rightarrow$ $x + 12 =$ speed of the car:
   $3.5x = 2.1(x + 12) \rightarrow 1.4x = 2.1 \cdot 12 \rightarrow x = 18$ $\rightarrow$ distance traveled = 63 miles.

2. Let the three amounts be $4x$, $3x$, and $2x$:
   $4x - 2000 = 2(5x + 2000) - 24000 \rightarrow 4x - 2000 = 10x - 20000$
   $\rightarrow 6x = 18000 \rightarrow 9x = 27000$

3. Let x = weight of the nuts in brand A's can $\rightarrow$ x = weight of the nuts in brand B's can.
   Let c = cost per pound of peanuts $\rightarrow$ $2c$ = cost per pound of walnuts and $4c$ = cost per pound of cashews.
   cost of nuts in brand A's can: $(.28x)(4c) + (.16x)(2c) + (.56x)(c) = 2xc$
   cost of nuts in brand B's can: $(.35x)(4c) + (.18x)(2c) + (.47x)(c) = 2.23xc$
   percent increase = $\frac{2.23 - 2}{2} = 0.115 = 11.5\%$

ROUND 3

1. $\sqrt{7\,{}^{1}\!/_{9}}-\sqrt{\frac{1}{9}+\frac{1}{16}}+\left(3+3^{-1}\right)^{2}=$

$\sqrt{\frac{64}{9}}-\sqrt{\frac{25}{9\cdot 16}}+\left(\frac{10}{3}\right)^{2}=\frac{8}{3}-\frac{5}{12}+\frac{100}{9}=\frac{96-15+400}{36}=\frac{481}{36}$

2. $\frac{\sqrt[3]{\sqrt[4]{x^{6}}}}{\sqrt{\sqrt[6]{x^{4}}}}=\sqrt{3}\cdot\sqrt[3]{2}\rightarrow\frac{x^{\frac{6}{12}}}{x^{\frac{4}{12}}}=3^{\frac{1}{2}}\cdot 2^{\frac{1}{3}}\rightarrow x^{\frac{1}{6}}=3^{\frac{1}{2}}\cdot 2^{\frac{1}{3}}\rightarrow x=3^{3}\cdot 2^{2}=108$

3. $\left(\frac{2^{-1}}{\sqrt[3]{2}-1}\right)\left(1-2^{1/6}\right)\left(1+2^{1/6}\right)-2^{-2}=$

$\left(\frac{2^{-1}}{\sqrt[3]{2}-1}\right)\left(1-2^{1/3}\right)-\frac{1}{4}=\left(\frac{2^{-1}}{\sqrt[3]{2}-1}\right)\left(1-\sqrt[3]{2}\right)-\frac{1}{4}=-\frac{1}{2}-\frac{1}{4}=-\frac{3}{4}$

ROUND 4

1. $x^{4}-x^{2}y^{2}-2xy^{3}+2y^{4}=$

$x^{3}\left(x^{2}-9\right)-8\left(x^{2}-9\right)=\left(x^{2}-9\right)\left(x^{3}-8\right)=\left(x-3\right)\left(x+3\right)\left(x-2\right)\left(x^{2}+2x+4\right)$

2. $2a^{2}+2b^{2}-5a+5b-4ab-12=$

$2a^{2}-4ab+2b^{2}-5a+5b-12=2\left(a-b\right)^{2}-5\left(a-b\right)-12=$

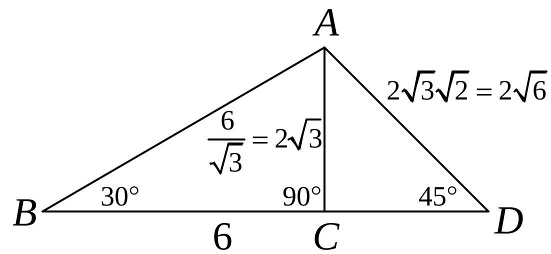
$\left(2\left(a-b\right)+3\right)\left(\left(a-b\right)-4\right)=\left(2a-2b+3\right)\left(a-b-4\right)$

3. $x^{2}-3a^{2}-xy-ay-2ax=x^{2}-2ax-3a^{2}-xy-ay=\left(x-3a\right)\left(x+a\right)-y\left(x+a\right)$

$=\left(x-3a-y\right)\left(x+a\right)$

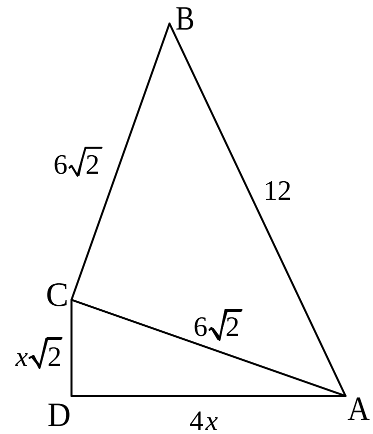
ROUND 5

1. $AC = \dfrac{6}{\sqrt{3}} = 2\sqrt{3} \rightarrow \; AD = 2\sqrt{3}\sqrt{2} = 2\sqrt{6}$



2. For the front wheel, $C = 12\pi$ in.; 315 rpm = $\dfrac{315 \cdot 12p \text{ in.}}{60\text{sec}} = 63p \text{ in./sec.}$

For the back wheel, $C = 3.5p \text{ ft.} = 42p \text{ in.} \; \dfrac{63p}{42p} = 1.5 \text{ rev/sec}$



3. $(4x)^2 + \left(x\sqrt{2}\right)^2 = \left(6\sqrt{2}\right)^2 \rightarrow 18x^2 = 72 \rightarrow x = 2 \;\rightarrow$

perimeter of ABCD = $12 + 6\sqrt{2} + 2\sqrt{2} + 8 = 20 + 8\sqrt{2}$.

TEAM ROUND

1. $462 \div 21 = 22$; $x = 21a,\; y = 21b,\; \text{LCM}(a,b) = 22$;

$a \neq 1 : (a,b) = (2,11),(2,22),(11,22)$; therefore the number of ordered pairs is 3.

2. In order for $4x^4 + ky^4$ to be factorable, when you complete its square the term being added and subtracted must itself be a perfect square , which is $2\left(2x^2\right)\left(\sqrt{k}\;y^2\right) \Rightarrow \sqrt{k}$ is a perfect square. Since k is odd and greater than 1, then the smallest value for k is 81.

$4x^4 + 81y^4 = 4x^4 + 36x^2y^2 + 81y^4 - 36x^2y^2 = \left(2x^2 + 9y^2\right)^2 - \left(6xy\right)^2 =$

$\left(2x^2 + 6xy + 9y^2\right)\left(2x^2 - 6xy + 9y^2\right)$

3. Let BC = a and AB = $c \rightarrow c^2 - a^2 = 105^2 = 3^2 \cdot 5^2 \cdot 7^2$; In order for $\sin(\angle \text{A})$ to be as small as possible, $c + a$ and $c - a$ must be factors of 105^2 and their difference must be non-zero, yet as small as possible; The largest factor of 105^2 less than 105 = 75;

$75 \times 147 = 105^2$; $\begin{cases} c + a = 147 \\ c - a = 75 \end{cases} \rightarrow c = 111,\; a = 36;\; \sin(\measuredangle \text{A}) = \dfrac{a}{c} = \dfrac{36}{111} = \dfrac{12}{37}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 2000

ANSWER SHEET:

ROUND 1

1. 1080
2. 50
3. $320_6, 410_6$ (320, 410)

ROUND 2

1. 63 (63 miles)
2. 27,000 ($27,000)
3. 11.5%

ROUND 3

1. $\frac{481}{36}$
2. 108
3. $-\frac{3}{4}$ (or equivalently -0.75)

ROUND 4

1. $(x-3)(x+3)(x-2)(x^2+2x+4)$
2. $(2a-2b+3)(a-b-4)$ or $(2b-2a-3)(b-a+4)$
3. $(x-3a-y)(x+a)$

ROUND 5

1. $2\sqrt{6}$
2. $\frac{3}{2}$ (equivalently $1\frac{1}{2}$ or 1.5)
3. $20+8\sqrt{2}$

TEAM ROUND

3 pts. 1. 3

3 pts. 2. $(2x^2+6xy+9y^2)(2x^2-6xy+9y^2)$

4 pts. 3. $\frac{12}{37}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2001

ROUND 1 – Arithmetic-Open

1. ____________%

2. ____________________

3. $(\_\_\_,\_\_\_)$

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. If a stock undergoes a 25% decrease, followed by a $46\frac{2}{3}\%$ decrease, followed by a 40% increase, from month to month over a three-month period, find the stock's percent decrease from its original price?

2. Find the sum of all odd 3-digit whole numbers which are divisible by 75.

3. Given $0.12_{(3)} - 0.13_{(4)} = 0.xy_{(12)}$. Find the ordered pair (x, y).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2001

ROUND 2 – Algebra 1 – Word Problems

1. ____________________

2. ____________________

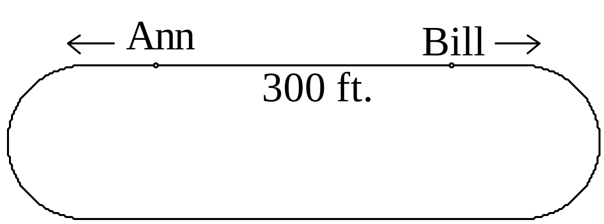
3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Kaitlin has three times as much money as her father. If she gives him $250, she would then have $20 less than twice his new amount. How many dollars did Kaitlin have originally?

2. A 20 liter acid solution is formed by mixing a certain amount of a 15% acid solution with an amount one-fifth more than that of a 25% acid solution and the rest a 30% acid solution. If this mixture is 25.8% acid, how many liters of the 15% acid are used in this solution?

3. Ann and Bill stand 300 feet apart on a track and run away from each other, in opposite directions. (See the diagram below.) They pass each other in 30 seconds. Ann completes one lap 60 seconds after they pass and Bill completes one lap 90 seconds after they pass. If Ann and Bill had a race one lap around this track, Ann would beat Bill by $\boldsymbol{f}$ feet. Solve for $\boldsymbol{f}$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2001

ROUND 3 – Algebra 1 – Exponents and Radicals

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Write the following expression in simplest radical form:

$$\left(\frac{\sqrt{2}}{1-\frac{\sqrt{2}}{\sqrt{3}}}\right)\left(\sqrt{6}-\frac{5}{\sqrt{6}}\right)$$

2. Find the value of the following expression:

$$\left(8^{-2/3}\right)\left(16^{-1/2}\right)+\left(2\tfrac{1}{4}\right)^{3/2}\left(\sqrt{3}\right)^{-2}$$

3. Solve the following equation for *x*:

$$\sqrt{4x+12}+11=x-\sqrt[4]{81x^2+486x+729}$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2001

ROUND 4 – Algebra 2– Factoring

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Factor the following completely: $12x^4 - 19x^2 - 18$

2. Factor the following completely: $4^x - 2^{x+3} + 2^4 - 9^x$

3. Factor the following completely: $3x^4 - 3x^3 - 102x^2 - 168x$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2001

ROUND 5 – Trigonometry: Angular and Linear Velocity; Right Triangle

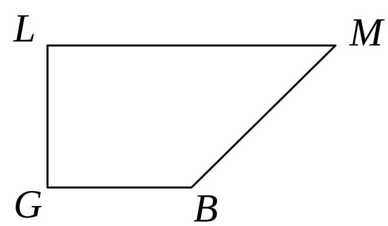
1. ______________________

2. ______________________

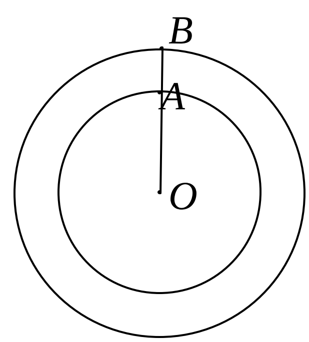
3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

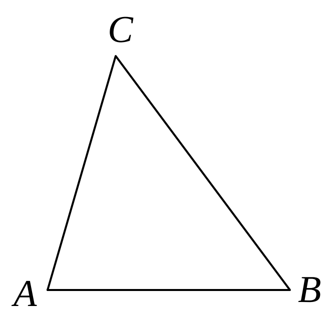
1. Given $GL = GB = 1$, $m\measuredangle G = 90°$, $m\measuredangle B = 135°$, and $\overline{GB} \parallel \overline{ML}$, find the perimeter of quadrilateral $GBML$.



2. Given concentric circles centered at point O with points A and B collinear with O, $AO = 6\text{cm}$ and $AB = 2\text{cm}$. A particle at A is rotating clockwise around the inner circle at $32p$ cm/sec and a particle at B is rotating clockwise around the outer circle at $30p$ cm/sec. What is total number of revolutions traveled by both particles the first time that they are back to this original position?



3. Given $\sin A = .96$, $\sin B = .8$, and the perimeter of $\Delta ABC = 4$, find the length of $\overline{AB}$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2001

TEAM ROUND (<u>12 MINUTES LONG</u>)

3 pts. 1. ____________________

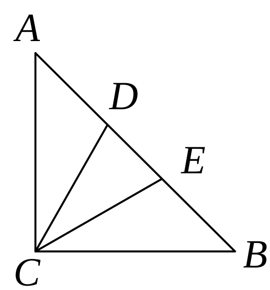
3 pts. 2. $(\_\_\_\_,\_\_\_\_)$

4 pts. 3.____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given n is a positive integer such that $2^n + 1$ is relatively prime with 15. Consider the set of all such n. Find the sum of the first 1000 elements in this set.

2. Given ΔABC is right isosceles with $\angle ACB$ right, $\overline{CD}$ and $\overline{CE}$ trisect $\angle ACB$, and $\dfrac{DE}{BC} = \sqrt{r} - \sqrt{s}$, find the ordered pair (r,s).



3. Find the area bounded by $9x^2 - 9xy + 9x + 2y^2 = 18$ and the positive coordinate axes.

Detailed Solutions for GBML MEET 1 – OCTOBER 2001

ROUND 1

1. $\frac{3}{4}\cdot\frac{8}{15}\cdot\frac{7}{5}=\frac{14}{25}=56\%\Rightarrow 44\%$ decrease.

2. Case I: last 2 digits 25: First : 2, 5, 8
   Case II: last 2 digits 75: First: 3, 6, 9 $\Rightarrow$ 225 + 525 + 825 + 375 + 675 + 975 = 3600.

3. $\left(\frac{1}{3}+\frac{2}{9}\right)-\left(\frac{1}{4}+\frac{3}{16}\right)=\frac{17}{144}=\frac{1}{12}+\frac{5}{144}\Rightarrow(x,y)=(1,5)$.

ROUND 2

1. Let x = Kaitlin's father's amount:
   $3x-250=2(x+250)-20\Rightarrow x=730\Rightarrow 3x=2190$

2. Let x = amount of the 15% acid solution:
   $.15x+.25\left(\frac{6}{5}x\right)+.30\left(20-\frac{11}{5}x\right)=.258(20)\Rightarrow .15x+.30x+6-.66x=5.16\Rightarrow$
   $.21x=.84\Rightarrow x=4$

3. Let x = number of feet in one lap $\Rightarrow$ $\frac{x}{90}=$Ann's speed and $\frac{x}{120}=$Bill's speed:
   $300+30\left(\frac{x}{90}\right)+30\left(\frac{x}{120}\right)=x\Rightarrow 300+\frac{7}{12}x=x\Rightarrow 300=\frac{5}{12}x\Rightarrow x=720$.
   Since Ann beats Bill by 30 seconds, the distance = $\frac{30}{120}(720)=180$ feet.

ROUND 3

1. $$\left(\frac{\sqrt{2}}{1-\frac{\sqrt{2}}{\sqrt{3}}}\right)\left(\sqrt{6}-\frac{5}{\sqrt{6}}\right)=\left(\frac{\sqrt{2}}{\frac{\sqrt{3}}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}}\right)\left(\frac{6}{\sqrt{6}}-\frac{5}{\sqrt{6}}\right)=\left(\frac{\sqrt{6}}{\sqrt{3}-\sqrt{2}}\right)\left(\frac{1}{\sqrt{6}}\right)=$$
$$\frac{1}{\sqrt{3}-\sqrt{2}}=\sqrt{3}+\sqrt{2}$$

2. $$\left(8^{-2/3}\right)\left(16^{-1/2}\right)+\left(2\tfrac{1}{4}\right)^{3/2}\left(\sqrt{3}\right)^{-2} = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)+\left(\frac{9}{4}\right)^{3/2}\left(\frac{1}{3}\right)=\frac{1}{16}+\left(\frac{27}{8}\right)\left(\frac{1}{3}\right)=\frac{1}{16}+\frac{9}{8}=\frac{19}{16}$$

3. $$\sqrt{4x+12}+11=x-\sqrt[4]{81x^2+486x+729} \Rightarrow \sqrt{4(x+3)}+11=x-\sqrt[4]{81\left(x^2+6x+9\right)}\Rightarrow$$
$$2\sqrt{x+3}+11=x-3\sqrt[4]{(x+3)^2} \Rightarrow 5\sqrt{x+3}=x-11\Rightarrow 25(x+3)=x^2-22x+121\Rightarrow$$
$$x^2-47x+46=0\Rightarrow (x-46)(x-1)=0 \Rightarrow x=46$$. ($x=1$ is an extraneous solution.)

ROUND 4

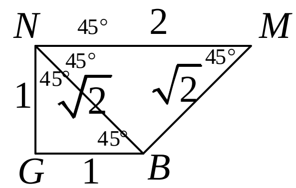
1. $$12x^4-19x^2-18=\left(4x^2-9\right)\left(3x^2+2\right)=(2x-3)(2x+3)\left(3x^2+2\right)$$

2. $$4^x-2^{x+3}+2^4-9^x = 2^{2x}-8\cdot 2^x+4^2-3^{2x}=\left(2^x-4\right)^2-3^{2x}=\left(2^x-3^x-4\right)\left(2^x+3^x-4\right)$$

3. $3x^4-3x^3-102x^2-168x = 3x\left(x^3-x^2-34x-56\right)$; now use synthetic division to factor the cubic polynomial: $\begin{array}{r|rrrr} 7 & 1 & -1 & -34 & -56 \\ \hline & 1 & 6 & 8 & 0 \end{array}$ $\Rightarrow$ the polynomial = $3x(x-7)\left(x^2+6x+8\right)=$
$$3x(x-7)(x+2)(x+4)$$

ROUND 5

1. Draw $\overline{BN}$. $BN=\sqrt{2} \Rightarrow BM=\sqrt{2} \Rightarrow MN=2 \Rightarrow$
Perimeter of $GBML = 4+\sqrt{2}$

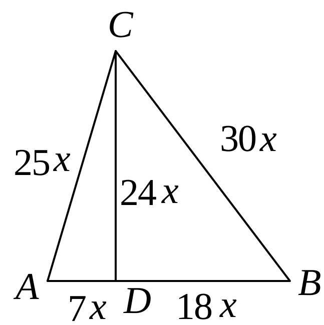


2. The angular velocity of the particle at $A = \dfrac{32p}{12p}\text{rev/sec} = \dfrac{8}{3}\text{rev/sec}$.

The angular velocity of the particle at $B = \dfrac{30p}{16p}\text{rev/sec} = \dfrac{15}{8}\text{rev/sec}$.

After 24 sec. particle at A traveled 64 revolutions and the particle at B traveled 45 revolutions for a total of 109 revolutions.

3. Draw altitude $\overline{CD}$. $\sin A = \dfrac{24}{25}$, $\sin B = \dfrac{4}{5} = \dfrac{24}{30}$; let $CD = 24x \Rightarrow$
$AC = 25x,\ AD = 7x,\ BC = 30x$, and $DB = 18x \Rightarrow$ perimeter =
$80x = 4 \Rightarrow x = 0.05$; $AB = 25x = 1.25$.



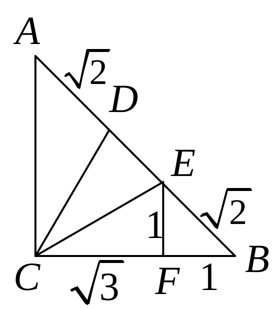
TEAM ROUND

1. $2^n + 1 \equiv (-1)^n + 1 \bmod 3 \Rightarrow$ all odd values for n are 0 mod3;
$2^{2n} + 1 = 4^n + 1 \equiv (-1)^n + 1 \bmod 5 \Rightarrow$ all even values for n that non-multiples of 4 are divisible by 5; the set is $\{4,8,12,16,...\}$; the sum of the first 1000 elements in this set is $500 \times 4004 = 2002000$

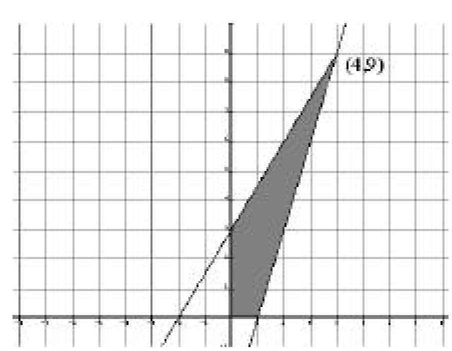
2. Draw $\overline{EF} \perp \overline{BC}$; since a ratio is required, there is no loss of generality to let $EF = 1 \Rightarrow BE = AD = \sqrt{2}$ and $CF = \sqrt{3}$;
therefore $BC = \sqrt{3}+1 \Rightarrow AB = \sqrt{2}\left(\sqrt{3}+1\right) = \sqrt{6}+\sqrt{2} \Rightarrow$

$$DE = \sqrt{6}-\sqrt{2} \Rightarrow \frac{DE}{BC} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{3}+1} = \frac{\left(\sqrt{6}-\sqrt{2}\right)\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)} =$$

$$\frac{4\sqrt{2}-2\sqrt{6}}{2} = 2\sqrt{2}-\sqrt{6} = \sqrt{8}-\sqrt{6} \Rightarrow (r,s) = (8,6).$$



3. $9x^2 - 9xy + 9x + 2y^2 = 18 \Rightarrow 9x^2 - 9xy + 9x + 2y^2 - 18 = 0$
$\Rightarrow (3x-2y+6)(3x-y-3) = 0 \Rightarrow 3x-2y+6=0$
and $3x-y-3=0$; these two lines intersect at $(4,9)$;
the first line has intercepts $(0,3)$ and $(-2,0)$;
the second line has intercept $(1,0)$;
the shaded area = $\dfrac{1}{2}\cdot 3 \cdot 9 - \dfrac{1}{2}\cdot 2 \cdot 3 = \dfrac{21}{2}$



GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2001

ANSWER SHEET:

ROUND 1

1. 44%
2. 3600
3. $(1,5)$

ROUND 2

1. 2190 ($2190)
2. 4 (4 liters)
3. 180 (180 feet)

ROUND 3

1. $\sqrt{3}+\sqrt{2}$
2. $\frac{19}{16}$
3. 46

ROUND 4

1. $(2x-3)(2x+3)(3x^2+2)$
2. $(2^x-3^x-4)(2^x+3^x-4)$
3. $3x(x-7)(x+2)(x+4)$

ROUND 5

1. $4+\sqrt{2}$
2. 109 (109 revolutions)
3. 1.25 or equivalent

TEAM ROUND

3 pts. 1. 2002000

3 pts. 2. $(8,6)$

4 pts. 3. 10.5 or equivalent

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2002

ROUND 1 – Arithmetic-Open

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. If $78_{(b)} = 67_{(b+2)}$, write this number in base 10.

2. The year 2002 is a palindrome. (It reads backwards the same as it reads forwards.). Let *S* be the sum of next three years that are palindromes. Write the prime factorization of *S*.

3. How many natural numbers less than 404 are divisible by 2, 3 or 4, and not by 6?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2002

ROUND 2 – Algebra 1 – Word Problems

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Joan's grandmother is 8 times as old as she is now. In 6 years her grandmother will be 5 times Joan's age then. Find the number of years old that Joan is now.

2. Kim invested a certain amount of money in Alpha Corporation and $2000 more than three times this amount in Beta Manufacturing. The percents profit Kim received from these two investments were 9% and 4%, respectively. If this was a 5.2% profit on the total amount invested, find the number of dollars invested in Alpha Corporation.

3. Jose rides a bicycle 4 feet per second faster downhill than on level ground and half his downhill rate when bicycling uphill. On a bicycle ride, the times he traveled on level ground, downhill, and uphill were in the ratio of 4:3:1 respectively. If Jose averaged 13 feet per second for this ride, and the distance he traveled downhill was 4800 feet, find the number of feet in the total distance of this bicycle ride.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2002

ROUND 3 – Algebra 1 – Exponents and Radicals

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Solve for n: $\sqrt{\frac{9}{16}+\frac{4}{25}}=\frac{3}{4}+\frac{2}{n}$

2. Put the following expression in simplest radical form:
$\left(\sqrt{1-4^{-2}}\right)\left(5^{-3/2}\right)\left(36^{-1/4}\right)$

3. When simplified, $\frac{\sqrt[3]{4}\cdot\sqrt[4]{3}}{\sqrt{2}}=\sqrt[n]{p}$, find the value of $\frac{p}{n}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2002

ROUND 4 – Algebra 2– Factoring

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Factor the following completely: $12x^2 + 23x - 24$.

2. Factor the following completely: $x^4 + x^3 + 12x - 144$.

3. The trinomial $x^2 + kx + 2002$, where k is a positive integer, is factorable over the integers. Find the smallest possible value for k.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2002

ROUND 5 – Trigonometry: Angular and Linear Velocity; Right Triangle

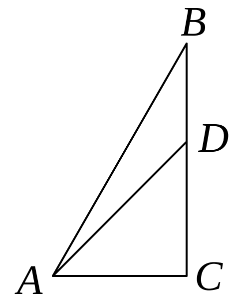
1. ______________________

2. ______________________

3. ______________________

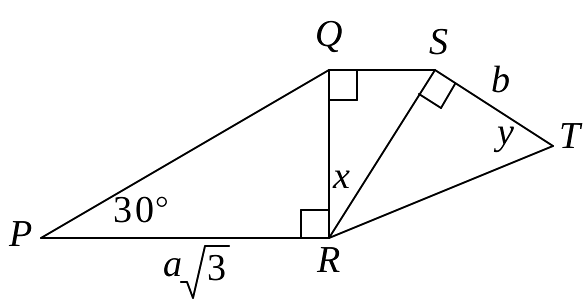
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. Given the figure on the right where $m\angle ACB = 90°$, $m\angle ADC = 45°$, $m\angle DAB = 15°$, and $BC = 6$, find the length of $\overline{AD}$.



2. A point on the rim of a merry-go-round making 20 revolutions every 3 minutes travels at a rate of 40*p* inches per second. Find the number of feet in the diameter of the merry-go-round.

3. Given the diagram on the right with the indicated measurements. Find *a* in terms of *b* and sines, cosines, **or** tangents of angles *x* and *y*.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2002

TEAM ROUND

3 pts. 1. ___________________

3 pts. 2. ___________________

4 pts. 3.____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

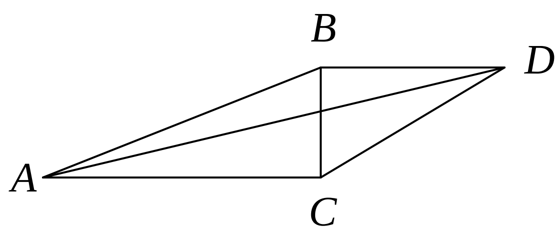
1. Solve the following equation over the real numbers: $\frac{1}{\sqrt[6]{x}} + 2\sqrt[3]{x} = \sqrt{x} \cdot \sqrt[3]{x}$

2. Find the sum of the five smallest whole numbers that have exactly 8 whole number factors.

3. Given $\overline{AC} \perp \overline{BC}$, $\overline{BD} \perp \overline{BC}$,

$\cos(\angle BAC) = \frac{5}{\sqrt{29}}$, and $\cos(\angle CDB) = \frac{5}{\sqrt{34}}$,

compute $\tan(\angle CAD)$.



Detailed Solutions for GBML MEET 1 – OCTOBER 2002

ROUND 1

1. $7b+8=6(b+2)+7 \Rightarrow 7b+8=6b+19 \Rightarrow b=11 \Rightarrow 7b+8=85$

2. The next three palindromes are 2112, 2222, and 2332 $\Rightarrow$ their sum is 6666. The prime factorization of 6666 = $2\times3\times11\times101$

3. The least common multiple of 2, 3, and 4 is 12. Of the numbers from 1 to 12, 2,3,4,8,9, and 10 are divisible by 2, 3, or 4 and not by 6. That means 6 out of 12 numbers satisfy the conditions. $404 \div 12 = 33$, remainder 8. From 1 to 8 are 3 out of the 6. To find the result: $33\times 6+3=201$.

ROUND 2

1. Let x = Joan's current age $\Rightarrow$ $8x$ = her grandmother's current age $\Rightarrow$
$8x+6=5(x+6) \Rightarrow 8x+6=5x+30 \Rightarrow 3x=24 \Rightarrow x=8$

2. Let x = amount invested in Alpha Corporation $\Rightarrow$ $3x + 2000$ = amount invested in Beta Manufacturing.
$.09x+.04(3x+2000)=.052(4x+2000) \Rightarrow .09x+.12x+80=.208x+104 \Rightarrow$
$.002x=24 \Rightarrow x=12000$

3. Let x = Jose's speed downhill $\Rightarrow$ $x-4=$ his speed on level ground and $\frac{1}{2}x=$ his speed uphill; let t = time bicycling uphill $\Rightarrow$ $3t=$ time bicycling downhill and $4t=$ time bicycling on level ground.
$\frac{1}{2}xt+3xt+4t(x-4)=13(8t) \Rightarrow \frac{15}{2}x-16=104 \Rightarrow x=16 \Rightarrow 48t=4800 \Rightarrow t=100 \Rightarrow$
total distance traveled on the bicycling trip = $13(800)=10400$ feet

ROUND 3

1. $\sqrt{\frac{9}{16}+\frac{4}{25}}=\frac{3}{4}+\frac{2}{n} \Rightarrow \sqrt{\frac{289}{16\cdot 25}}=\frac{3}{4}+\frac{2}{n}\Rightarrow\frac{17}{20}=\frac{15}{20}+\frac{2}{n}\Rightarrow\frac{2}{20}=\frac{2}{n}\Rightarrow n=20$

2. $\left(\sqrt{1-4^{-2}}\right)\left(5^{-3/2}\right)\left(36^{-1/4}\right)=\left(\sqrt{\frac{15}{16}}\right)\left(\frac{1}{5\sqrt{5}}\right)\left(\left(6^2\right)^{-1/4}\right)=\frac{\sqrt{15}}{4}\left(\frac{1}{5\sqrt{5}}\right)\left(6^{-1/2}\right)=$
$\frac{\sqrt{15}}{4}\left(\frac{1}{5\sqrt{5}}\right)\left(\frac{1}{\sqrt{6}}\right)=\frac{1}{20\sqrt{2}}=\frac{\sqrt{2}}{40}$

3. $\frac{\sqrt[3]{4}\cdot\sqrt[4]{3}}{\sqrt{2}}=\frac{2^{2/3}\cdot 3^{1/4}}{2^{1/2}}=2^{1/6}\cdot 3^{1/4}=2^{2/12}\cdot 3^{3/12}=\sqrt[12]{2^2\cdot 3^3}\Rightarrow\frac{p}{n}=\frac{2^2\cdot 3^3}{12}=9.$

ROUND 4

1. $12x^2+23x-24=(4x-3)(3x+8)$.

2. $x^4+x^3+12x-144 = x^4-144+x^3+12x=\left(x^2-12\right)\left(x^2+12\right)+x\left(x^2+12\right) =$
$\left(x^2+12\right)\left(x^2+x-12\right)=\left(x^2+12\right)(x+4)(x-3)$

3. 2002 = 2×7×11×13; to find the smallest value for *k* find two numbers that multiply to 2002 whose sum is a minimum; the two numbers are 2×13 = 26 and 7×11 = 77 ⇒ the smallest value of *k* is 26 + 77 = 103.

ROUND 5

1. Since ΔABC is a 30-60-90° triangle $\Rightarrow AC = \dfrac{6}{\sqrt{3}} = 2\sqrt{3} \Rightarrow AD = 2\sqrt{3}\sqrt{2} = 2\sqrt{6}$.

2. 20 revolutions/ 3 minutes = 20 revolutions/ 180 seconds = 1/9 revolution/ second = $40p$ inches/ second $\Rightarrow$ 1 revolution = $9 \times 40p$ inches $\Rightarrow$ diameter = 9×40 inches = 30 feet.

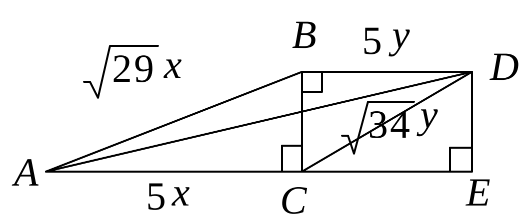
3. $QR = a,\ \cos x = \dfrac{a}{RS} \Rightarrow RS = \dfrac{a}{\cos x};\ \tan y = \dfrac{RS}{b} \Rightarrow RS = b\tan y \Rightarrow$
$\dfrac{a}{\cos x} = b\tan y \Rightarrow a = b\cos x \tan y$.

TEAM ROUND

1. $\dfrac{1}{\sqrt[6]{x}} + 2\sqrt[3]{x} = \sqrt{x} \cdot \sqrt[3]{x} \Rightarrow 1 + 2\sqrt{x} = x \Rightarrow 2\sqrt{x} = x - 1 \Rightarrow 4x = x^2 - 2x + 1 \Rightarrow x^2 - 6x + 9 = 8$
$\Rightarrow (x-3)^2 = 8 \Rightarrow x - 3 = \pm 2\sqrt{2} \Rightarrow x = 3 \pm 2\sqrt{2}$; $3 - 2\sqrt{2}$ is extraneous so the only solution is $3 + 2\sqrt{2}$.

2. A number with 8 factors is of the forms $p \cdot q \cdot r,\ p^3 q,$ or p^7, where p, q, and r are primes. $2^3 \cdot 3 = 24,\ 2 \cdot 3 \cdot 5 = 30,\ 2^3 \cdot 5 = 40,\ 2 \cdot 3 \cdot 7 = 42,\ 2 \cdot 3^3 = 54$ are the five smallest. Their sum = 190.

3. Extend $\overline{AC}$ and draw a perpendicular from D intersecting at E. $BC = 2x = 3y$; Let $BC = 6 \Rightarrow$
$DE = 6$ and $AE = 25 \Rightarrow \tan(\angle CAD) = \dfrac{6}{25} = 0.24$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2002

ANSWER SHEET:

ROUND 1

1. 85
2. $2\times3\times11\times101$
3. 201

ROUND 2

1. 8 (8 years old)
2. 12,000 ($12,000)
3. 10400 (10400 ft.)

ROUND 3

1. 20
2. $\dfrac{\sqrt{2}}{40}$
3. 9

ROUND 4

1. $(4x-3)(3x+8)$
2. $(x+4)(x-3)(x^2+12)$
3. 103

ROUND 5

1. $2\sqrt{6}$
2. 30 (30 ft.)
3. $b\cos x\tan y \quad \left(\text{or } \dfrac{b\cos x\sin y}{\cos y}\right)$

TEAM ROUND

3 pts. 1. $3+2\sqrt{2}$

3 pts. 2. 190

4 pts. 3. $\dfrac{6}{25}$ (or 0.24)

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2003

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given digit X is less than digit Y, how many possibilities are there for the two-digit number XY if XY and YX are both two-digit prime numbers?

2. If the product of two whole numbers is 132, what is the smallest possible value for the sum of their squares?

3. If $RS_{(5)} = SR_{(9)}$ and R and S are non-zero digits, find all possible numbers that satisfy this equation expressed in base 7.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2003

ROUND 2 – Algebra 1 – Word Problems

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Two positive numbers are in the ratio of 7:3. If 2 is subtracted from each, the larger is now 6 more than twice the smaller. Find the sum of the original two numbers.

2. Find all two digit natural numbers satisfying the condition that 5 times its ten's digit is 1 less than 4 times it unit's digit.

3. In a triathlon, Katie swam at 3 mph, ran at 9 mph and biked at 18 mph. The distance she ran equaled the distance she biked and the triathlon took her 4 hours to complete a total of 33 miles. How many miles did Katie swim?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2003

ROUND 3 – Algebra 1 – Exponents and Radicals

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Simplify the following: $20\left(\frac{\sqrt{5}}{\sqrt{12}}-\frac{\sqrt{3}}{\sqrt{45}}\right)$

2. Solve the following equation over the real numbers: $\sqrt[3]{x^2}=9\sqrt[3]{x}-8$

3. Given positive integers *a*, *b*, and *c*, such that $\sqrt{a}+\sqrt{b}=\sqrt{c}$. If $a=63$ and $c\le 500$, find the largest possible value for *b*.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2003

ROUND 4 – Algebra 2– Factoring

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Factor the following completely: $17x^2 - 20x + 10x^3$.

2. Factor the following completely: $x^4 - ax^3 - a^3x + a^4$.

3. Factor the following completely: $2x^5y + 6x^3y^3 + 8xy^5$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2003

ROUND 5 – Right Triangle Trigonometry, Angular and Linear Velocity

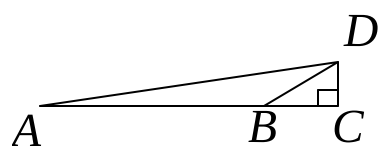
1. ____________________

2. ____________________ feet per minute

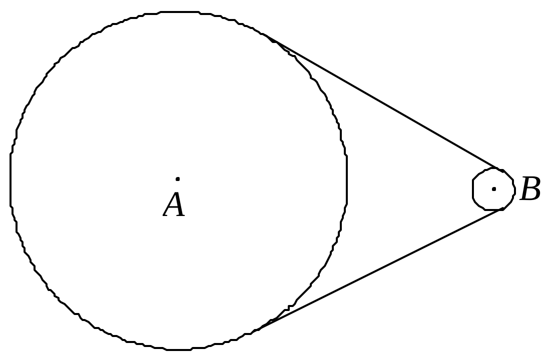
3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

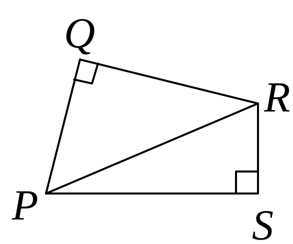
1. Given $m\angle C = 90°$, $m\angle DBC = 30°$ and $AB : BC = 3:1$. Find the value of the sine of $\angle A$.



2. A drive belt is attached to circles *A* and *B*. Circle *A* with a radius of 2 feet is revolving at a rate of $60p$ feet per second. If circle *B* has a radius of 3 inches, find how fast any point on circle *B* is revolving in feet per minute.



3. Given right angles *Q* and *S*, $\tan\angle QPR = \frac{4}{3}$, and $\tan\angle SPR = \frac{5}{12}$, find the value of $\frac{PQ+RS}{QR+PS}$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2003

TEAM ROUND Time limit: 12 minutes

3 pts. 1. __________________

3 pts. 2. __________________

4 pts. 3.___________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given five numbers, which when listed in order, satisfy the condition that the difference between every two consecutive numbers equals 7. These numbers also satisfy the condition that the sum of three of them equals the sum of the remaining two. Find the difference between the largest and smallest possible sum that these five numbers can have.

2. Three numbers are in the ratio of 3:7:8. Five times the largest is thirty more than twice the sum of the two smallest. Find all possible sums of these three numbers.

3. The number $(5!+6!+7!)^3$ has how many perfect square factors?

Detailed Solutions for GBML MEET 1 – OCTOBER 2003

ROUND 1

1. 13, 17, 37, 79 are the only ones; therefore, the answer is 4.

2. The two whole numbers you are looking for are the pair of factors of 132 with their difference a minimum. Those would be 11 and 12. $11^2 + 12^2 = 265$. The reason why this is true is that the pair of factors with the smallest sum is the one with the smallest difference and since $(a+b)^2 = a^2 + b^2 + 2ab = a^2 + b^2 + 2(132)$ in this case; $a+b$ a minimum $\Rightarrow a^2 + b^2$ is also a minimum.

3. $5R + S = 9S + R \Rightarrow 4R = 8S \Rightarrow R = 2S \Rightarrow 21_{(5)}$ or $42_{(5)}$ are the only possibilities; these numbers = 11 or 22 (base 10) = $14_{(7)}, 31_{(7)}$.

ROUND 2

1. Let the two numbers be $7x$ and $3x \Rightarrow 7x - 2 = 2(3x-2) + 6 \Rightarrow x = 4 \Rightarrow 10x = 40$.

2. Let t = its ten's digit and u = its unit's digit $\Rightarrow 5t = 4u - 1$; this equation is only true when $u = 4,\ t = 3$ and when $u = 9, t = 7$; therefore the only two digit numbers are 34 and 79.

3. Since Katie's biking speed is twice her running speed and the distances are equal $\Rightarrow$ if t = time biking, then $2t$ = time running $\Rightarrow 4 - 3t =$ time swimming $\Rightarrow$ $18t + 18t + 3(4-3t) = 33 \Rightarrow 27t = 21 \Rightarrow t = \frac{7}{9} \Rightarrow 3\left(4 - 3\left(\frac{7}{9}\right)\right) = 5$ miles.

ROUND 3

1. $20\left(\frac{\sqrt{5}}{\sqrt{12}}-\frac{\sqrt{3}}{\sqrt{45}}\right) = 20\left(\frac{\sqrt{5}}{2\sqrt{3}}-\frac{\sqrt{3}}{3\sqrt{5}}\right)=20\left(\frac{15-6}{6\sqrt{15}}\right)=\frac{180}{6\sqrt{15}}=\frac{30}{\sqrt{15}}=\frac{30\sqrt{15}}{15}=2\sqrt{15}$

2. $\sqrt[3]{x^2}=9\sqrt[3]{x}-8 \Rightarrow \sqrt[3]{x^2}-9\sqrt[3]{x}+8=0 \Rightarrow \left(\sqrt[3]{x}-1\right)\left(\sqrt[3]{x}-8\right)=0 \Rightarrow \sqrt[3]{x}=1,8 \Rightarrow x=1,512$

3. Since $63=9\times 7$, b and c must also be 7 times a perfect square; the largest multiple of 7 less than 500 is $71\times 7 \Rightarrow c=7\times 64 \Rightarrow b=7\times 25=175$.

ROUND 4

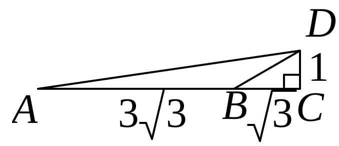
1. $17x^2-20x+10x^3 = 10x^3+17x^2-20x = x\left(10x^2+17x-20\right)=x\left(5x-4\right)\left(2x+5\right)$

2. $x^4-ax^3-a^3x+a^4 = x^3\left(x-a\right)-a^3\left(x-a\right)=\left(x-a\right)\left(x^3-a^3\right)=\left(x-a\right)^2\left(x^2+ax+a^2\right)$

3. $2x^5y+6x^3y^3+8xy^5 = 2xy\left(x^4+3x^2y^2+4y^4\right)=2xy\left(x^4+4x^2y^2+4y^4-x^2y^2\right)=$
$2xy\left(\left(x^2+2y^2\right)^2-\left(xy\right)^2\right)=2xy\left(x^2-xy+2y^2\right)\left(x^2+xy+2y^2\right)$

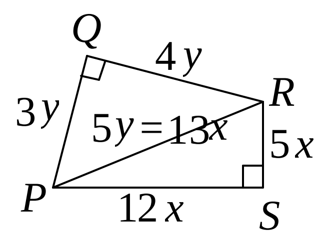
ROUND 5

1. Since ΔBCD is a 30-60-90° triangle $\Rightarrow$ if $CD = 1 \Rightarrow BC = \sqrt{3} \Rightarrow AB = 3\sqrt{3} \Rightarrow AC = 4\sqrt{3} \Rightarrow$ $AD = \sqrt{1+48} = 7 \Rightarrow \sin\angle A = \frac{1}{7}$.



2. Circle A has a circumference of $4p$ feet $\Rightarrow$ Circle A is rotating at $\frac{60p}{4p} = 15$ revolutions per second = 900 revolutions per minute; since circle A has a circumference 8 times circle $B \Rightarrow$ circle B is rotating 7200 revolutions per minute and since the circumference of circle B = $\frac{1}{2}p$ feet $\Rightarrow$ a point on circle B is traveling at 3600p feet per minute.

3. Let $RS = 5x$, $PS = 12x \Rightarrow PR = 13x$; let $QR = 4y$, $PQ = 3y \Rightarrow PR = 5y$; $13x = 5y$ is true if $x = 5$ and $y = 13 \Rightarrow \frac{PQ+RS}{QR+PS} = \frac{39+25}{52+60} = \frac{64}{112} = \frac{4}{7}$



TEAM ROUND

1. The five numbers are $x,\ x+7,\ x+14,\ x+21,\ x+28$.
To find the largest possible sum: $3x+21 = 2x+49 \Rightarrow x = 28$.
To find the smallest possible sum: $3x+63 = 2x+7 \Rightarrow x = -56$.
The sum of the five numbers = $5x+70 \Rightarrow (5(28)+70) - (5(-56)+70) = 5(84) = 420$.

2. Let the three numbers be $3x,\ 7x,$ and $8x \Rightarrow$ if $x > 0$, then $40x = 20x+30 \Rightarrow x = \frac{3}{2}$ and if $x < 0$, then $15x = 30x+30 \Rightarrow x = -2$; the sum of the three numbers = $18x = 27$ or -36.

3. $(5!+6!+7!)^3 = (5!(1+6+42))^3 = (5!\times 49)^3 = (2^3\times 3\times 5\times 7^2)^3 = 2^9\times 3^3\times 5^3\times 7^6$; a perfect square factor of this number will have 0, 2, 4, 6, or 8 factors of 2, 0 or 2 factors of 3, 0 or 2 factors of 5, and 0, 2, 4, or 6 factors of 7 $\Rightarrow$ using the basic counting principle, the number of perfect square factors = $5\times 2\times 2\times 4 = 80$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2003

ANSWER KEY:

ROUND 1

1. 4
2. 265
3. $14_{(7)}, 31_{(7)}$ (or 14, 31)

ROUND 2

1. 40
2. 34, 79
3. 5 (5 miles)

ROUND 3

1. $2\sqrt{15}$
2. 1, 512
3. 175

ROUND 4

1. $x(5x-4)(2x+5)$
2. $(x-a)^2(x^2+ax+a^2)$
3. $2xy(x^2-xy+2y^2)(x^2+xy+2y^2)$

ROUND 5

1. $\frac{1}{7}$
2. $3600p$ feet per minute
3. $\frac{4}{7}$

TEAM ROUND

3 pts. 1. 420

3 pts. 2. 27, –36

4 pts. 3. 80

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1998

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the following summation in base 6, $b34_6 + aac_6 = a0ba_6$, find the sum, $a + b + c$ **in base 6** .

2. How many perfect square factors does 4,000,000 have?

3. Find the 1998th counting number divisible by 4 but not by 5.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1998

ROUND 2 – Simultaneous Linear Equations, Word Problems, Matrices

1. (______ , ______)

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the ordered pair, (x, y), solution to the following system of equations:

$$\begin{cases} \frac{x}{2} - \frac{y}{3} = 4 \\ \frac{x - 4y}{14} = 2 \end{cases}$$

2. Given the matrix equation, $\begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ -8 & 9 \end{pmatrix} = \begin{pmatrix} 5y + 16 & 2x - 3 \\ a & b \end{pmatrix}$,

find the sum $a + b + x + y$.

3. If sixty coins consisting of nickels, dimes, and quarters are worth exactly five dollars and fifteen cents, what is the most number of quarters you can have ?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1998

ROUND 3 – Geometry: Angles and Triangles

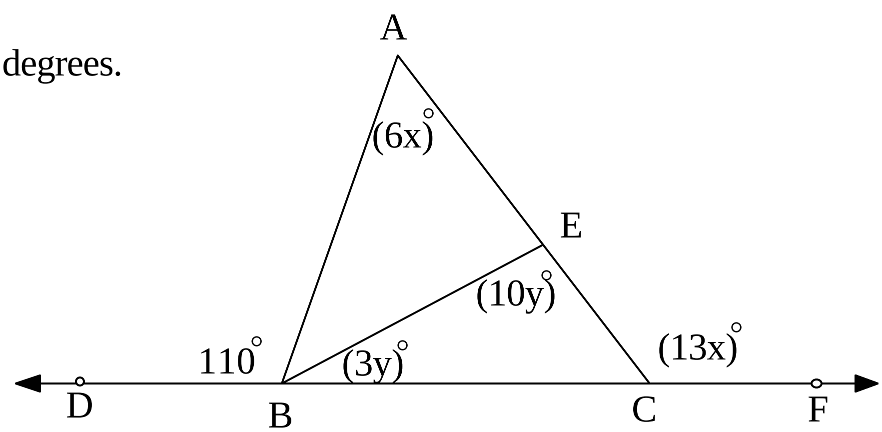
1. ____________________

2. ____________________

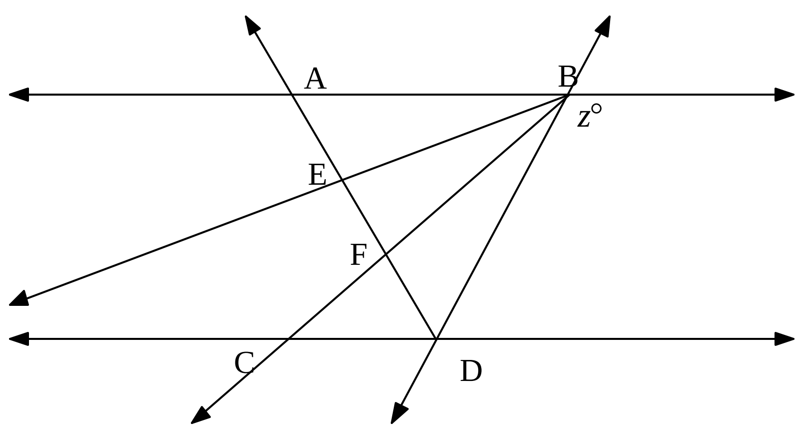
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

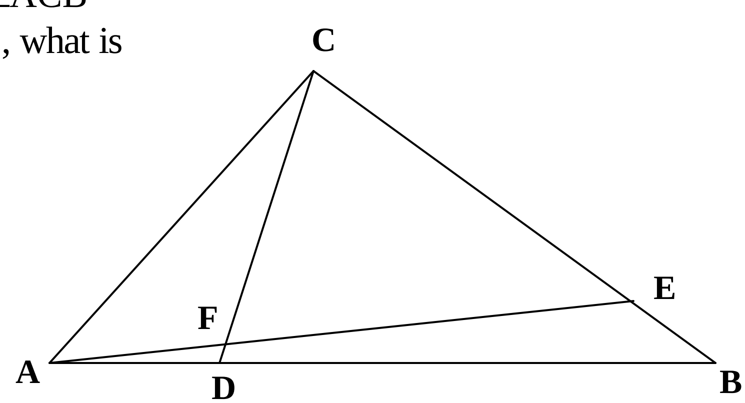
1. Find the measure of $\angle$ AEB in degrees.



2. If $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$, $\overrightarrow{BE}$ and $\overrightarrow{BF}$ trisects $\angle$ABD, $\overrightarrow{DA}$ bisects $\angle$CDB, and m $\angle$BED = 77°, find *z*.



3. Given m $\angle$CBA : m $\angle$BAC : m $\angle$ACB = 3:4:8, $\overline{BD} \cong \overline{BC}$, and $\overline{CA} \cong \overline{CE}$, what is measure of $\angle$ AFC in degrees?



GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1998

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all solutions for real x for the following equation: $\frac{3}{x-3} - \frac{x^2+3}{4x-12} = \frac{x-2}{4}$

2. Given a quadratic equation in standard form with leading coefficient equaling 1, with roots 2 and r, and the value of its discriminant is 25, find all solutions for r.

3. Kaitlin's one hundred mile road trip by bicycle included exactly twenty-five miles uphill, fifteen miles downhill, and the rest on level ground. If her speed downhill was three times her speed uphill and her speed on level ground was eight miles per hour faster than her speed uphill, find her speed uphill in miles per hour if her travelling time totaled five hours and thirty minutes.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1998

ROUND 5 – Trig. Equations

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given $\mathbf{0^\circ \le \theta < 360^\circ}$, solve the following equation for θ:
$\tan\theta \cdot \sin\theta + \cos\theta = \sec\theta$

2. Given $\cos 2x = \tan^2 x$, find all values for $\mathbf{\cos x}$ in simplified radical form.

3. Given $\mathbf{0^\circ \le \theta \le 180^\circ}$, solve the following equation for θ: $\sin\theta + \cos\theta = \frac{\sqrt{6}}{2}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1998

TEAM ROUND

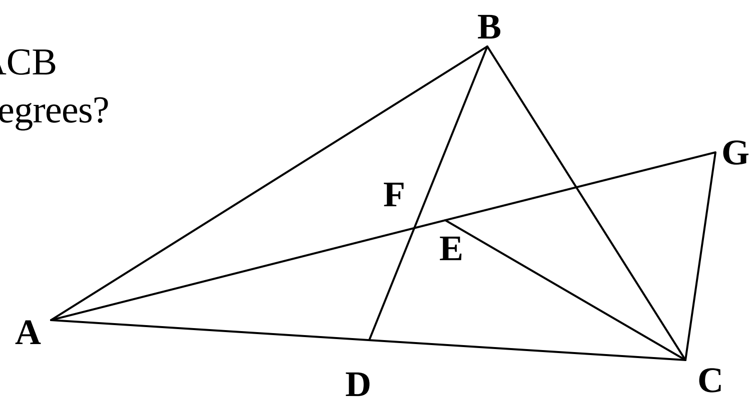
3 pts. 1. ___________________

3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND

1. Given AD = DC = BD, EC = EG, $m\angle AFB = 100° + m\angle DCE$, and $\overline{AG}$ and $\overline{CE}$ bisect $\angle BAC$ and $\angle ACB$ respectively, what is $m\angle BCG$ in degrees?



2. The five digit base ten number, 9*x*3*y*6, is divisible by 36. How many different ordered pairs (x,y) satisfy this condition?

3. Given the equation in *x*, $kx^2 + p + 6kx - 3x^2 = 0$, with real numbers *k* and *p*. Find all ordered pairs (k, p) for which the sum of the roots of the equation will be $2k + 3$ and the roots are real and equal.

Detailed Solutions of GBML MEET 2 – NOVEMBER 1998

ROUND 1

1. $b34_6 + aac_6 = a0ba_6 \Rightarrow a = 1 \Rightarrow b34_6 + 11c_6 = 10b1_6 \Rightarrow 4 + c = 11_6 \Rightarrow c = 3 \Rightarrow$
$b34_6 + 113_6 = 10b1_6 \Rightarrow 1 + 3 + 1 = c \Rightarrow c = 5 \Rightarrow a + b + c = 9 = 13_6$

2. $4{,}000{,}000 = 2^8 \cdot 5^6$. The perfect square factors of 4,000,000 have an even power of 2 and an even power of 5. The even powers of 2 are 0,2,4,6, and 8. The even powers of 5 are 0,2,4, and 6. $\Rightarrow$There are $5 \cdot 4 =$ **20** factors that are perfect squares.

3. Consider the set of counting numbers from 1 to 20. Of these only 4, 8, 12, and 16 are divisible by 4, but not by 5; i.e. 4 out of every 20. $1998 \div 4 = 499$ R 2; 8 is the 2nd number divisible by 4 and not by 5 $\Rightarrow$ answer is $499 \times 20 + 8 =$ **9988**.

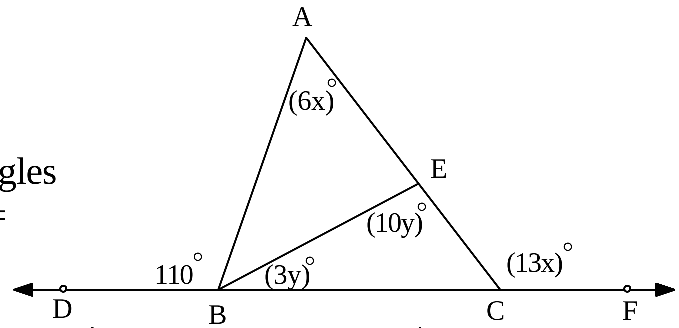
ROUND 2

1. $\begin{cases} \frac{x}{2} - \frac{y}{3} = 4 \\ \frac{x - 4y}{14} = 2 \end{cases} \Rightarrow \begin{cases} 3x - 2y = 24 \\ x - 4y = 28 \end{cases} \Rightarrow \begin{cases} -6x + 4y = -48 \\ x - 4y = 28 \end{cases} \Rightarrow -5x = -20 \Rightarrow x = 4 \text{ and } y = -6$
$\Rightarrow$ **(4, -6)** is the answer.

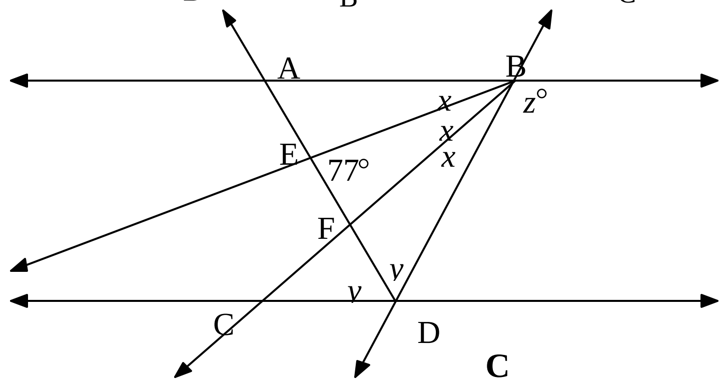
2. $\begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ -8 & 9 \end{pmatrix} = \begin{pmatrix} 5y + 16 & 2x - 3 \\ a & b \end{pmatrix} \Rightarrow 6x + 8 = 5y + 16 \text{ and } 6y - 9 = 2x - 3 \Rightarrow$
$6x - 5y = 8$ and $-2x + 6y = 6 \Rightarrow x = 3$ and $y = 2 \Rightarrow a = -17$ and $b = 46 \Rightarrow$
$a + b + x + y =$ **34**

3. Let n = number of nickels, d = number of dimes, and q = number of quarters:
$n + d + q = 60$ and $5n + 10d + 25q = 515 \Rightarrow n + 2d + 5q = 103 \Rightarrow d + 4q = 43$
To find the maximum number of quarters let $d = 3 \Rightarrow q =$ **10**

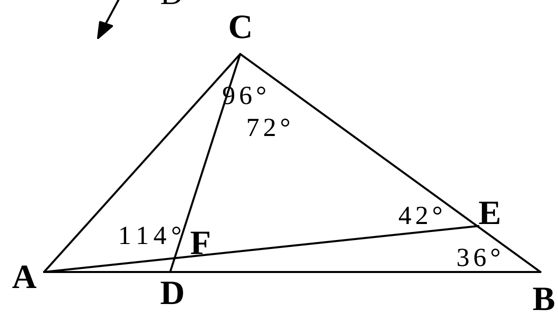
ROUND 3



1. $13x = 3y + 10y$ Þ $x = y$ Also,
$13x + 110 + 180 - 6x = 360$ (Sum of ext. angles $= 360°$) $\Rightarrow x = 10 \Rightarrow y = 10 \Rightarrow$ m $\angle$ AEB = $180° - 100°$ = **80°**



2. Because of the parallel lines:
$3x + 2y = 180°$ and because of **D** BED, $2x + y = 103° \Rightarrow$
$x = 26°$ and $z = 180 - 78 =$ **102**



3. $3x + 4x + 8x = 180° \Rightarrow x = 12° \Rightarrow$
36°, 48°, and 96° are the angles of the triangle. $\Rightarrow$
m $\angle$ AEC = $(180° - 96°) \div 2 = 42°$
and m $\angle$ BCD = $(180° - 36°) \div 2 = 72° \Rightarrow$
m $\angle$ AFC = $42° + 72°$ = **114°**

ROUND 4

1. $\frac{3}{x-3} - \frac{x^2+3}{4x-12} = \frac{x-2}{4} \Rightarrow x \neq 3$ and $12 - (x^2+3) = (x-3)(x-2) \Rightarrow$
$x \neq 3$ and $12 - x^2 - 3 = x^2 - 5x + 6 \Rightarrow 2x^2 - 5x - 3 = 0 \Rightarrow (2x+1)(x-3) = 0 \Rightarrow x = \mathbf{-\frac{1}{2}}$

2. Since a, the coefficient of x^2, is 1, and the discriminant is 25, the roots of the quadratic equation are $\frac{-b \pm 5}{2}$, where b is the coefficient of x. $\Rightarrow$ Difference of the roots is 5 $\Rightarrow$ the second root is either 2 + 5 or 2 – 5. $\Rightarrow$ The second root is **–3 or 7.**

3.

	speed	distance	time
uphill	x	25	$25/x$
downhill	$3x$	15	$5/x$
level	$x + 8$	60	$60/(x + 8)$

Equation: $\frac{30}{x} + \frac{60}{x+8} = \frac{11}{2} \Rightarrow 60(x+8) + 120x = 11x(x+8) \Rightarrow 11x^2 - 92x - 480 = 0$
$\Rightarrow (11x + 40)(x - 12) = 0 \Rightarrow x =$ **12** mph

ROUND 5

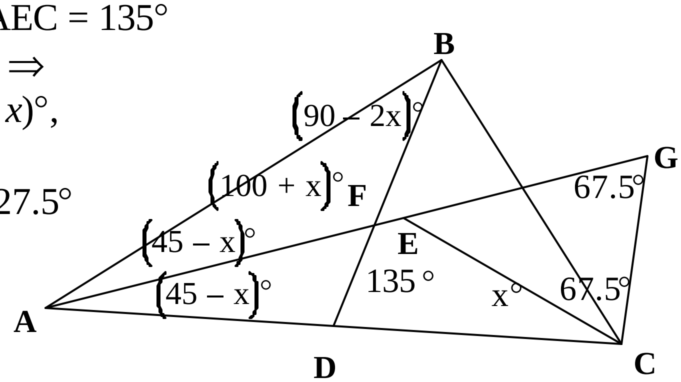
1. $\dfrac{\sin^2\theta}{\cos\theta} + \dfrac{\cos\theta}{1} = \dfrac{1}{\cos\theta}$, $\cos\theta \neq 0 \Rightarrow \dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \dfrac{1}{\cos\theta}$, which is an identity $\Rightarrow$ The equation is always true unless $\cos\theta = 0 \Rightarrow$ **$\theta \neq 90°$ and $\theta \neq 270°$**

2. $\cos 2x = \tan^2 x \Rightarrow 2\cos^2 x - 1 = \sec^2 x - 1 \Rightarrow 2\cos^4 x = 1 \Rightarrow \cos x = \pm\dfrac{1}{\sqrt[4]{2}} = \pm\dfrac{\sqrt[4]{8}}{2}$

3. $\sin\theta + \cos\theta = \dfrac{\sqrt{6}}{2} \Rightarrow (\sin\theta + \cos\theta)^2 = \left(\dfrac{\sqrt{6}}{2}\right)^2 \Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cdot\cos\theta = \dfrac{3}{2}$
$\Rightarrow 2\sin\theta\cdot\cos\theta = \dfrac{1}{2} \Rightarrow \sin 2\theta = \dfrac{1}{2}$; Since $0° \le \theta \le 180°$, then $0° \le 2\theta \le 360° \Rightarrow$
$2\theta = 30°$ or $150° \Rightarrow \theta =$ **15°** or **75°** (which both check into the original equation)

TEAM ROUND

1. Since AD = BD = CD, Δ ABC is right and because of the angle bisectors, m $\angle$AEC = 135° and m $\angle$ECG = 67.5°. Call m ACE = x° $\Rightarrow$ m $\angle$AFB = $(100 + x)$°, m $\angle$BAF = $(45 - x)$°, and m $\angle$ABD = $(90 - 2x)$°; Equation: $100 + x + 45 - x + 90 - 2x = 180 \Rightarrow x = 27.5°$
$\Rightarrow$ m $\angle$BCG = 67.5° – 27.5° = **40°**



2. In order for 9x3y6 to be divisible by 36, it must be divisible by 4 and 9. Divisibility by 4 $\Rightarrow y$ = 1, 3, 5, 7, or 9. (4 divides evenly into the last 2 digits.) Now consider each case: $y = 1 \Rightarrow x = 8$; (The sum of the digits is divisible by 9.) $y = 3 \Rightarrow x = 6$; $y = 5 \Rightarrow x = 4$; $y = 7 \Rightarrow x = 2$; $y = 9 \Rightarrow x = 0$ or 9; $\Rightarrow$ There are **6** possible ordered pairs.

3. $kx^2 + p + 6kx - 3x^2 = 0 \Rightarrow (k-3)x^2 + 6kx + p = 0 \Rightarrow$ sum of the roots $= \dfrac{-6k}{k-3} = 2k + 3$
$2k^2 + 3k - 9 = 0 \Rightarrow (2k-3)(k+3) = 0 \Rightarrow k = -3,\ 1.5$; Two equal roots $\Rightarrow$
$(6k)^2 - 4(k-3)p = 0$; If k = - 3, then $(-18)^2 - 4(-6)p = 0 \Rightarrow p$ = -13.5
If k = 1.5, then $(9)^2 - 4(-1.5)p = 0 \Rightarrow p$ = -13.5 $\Rightarrow$ **(-3, -13.5) and (1.5, -13.5)** are the ordered pairs.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1998

ANSWER SHEET:

ROUND 1

1. 13_6 (13 is acceptable.)
2. 20
3. 9988

ROUND 2

1. (4, -6)
2. 34
3. 10

ROUND 3

1. 80 (80°)
2. 102 (102°)
3. 114 (114°)

ROUND 4

1. $-\frac{1}{2}$
2. –3, 7
3. 12

ROUND 5

1. $\theta \neq 90°$ and $\theta \neq 270°$ (and $0° \leq \theta < 360°$)

Note: In problem 1, the domain is not necessary.

2. $\pm \frac{\sqrt[4]{8}}{2}$
3. 15°, 75°

TEAM ROUND

3 pts. 1. 40°

3 pts. 2. 6

4 pts. 3. (-3, -13.5) and (1.5, -13.5)

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1999

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. If n % of 55 is 25% of 88, find n.

2. Given that the following multiplication of two digit base ten numbers produces a result which is divisible by nine, compute $x+y+z$.
Note that x, y, and z are not necessarily distinct digits.

$$\begin{array}{ccc} & 2 & x \\ & 4 & y \\ \hline 9 & 4 & z \end{array}$$

3. If $\sqrt{\sqrt{6!a}}$ is a positive integer, find the smallest possible integral value for a.
Note that $n!=n(n-1)(n-2)....(2)(1)$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1999

ROUND 2 – Simultaneous Linear Equations, Word Problems, Matrices

1. ____________________

2. (______ , ______)

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the following system of equations in *x* and *y*, find *x* + *y* in terms of *a*:

$$\begin{cases} 3x + 4y = -a \\ 9x - 8y = 7a \end{cases}$$

2. Find the ordered pair, (*x*, *y*), solution to the following matrix equation:

$$\begin{pmatrix} x & y \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & y-4 \\ 2 & x \end{pmatrix} = \begin{pmatrix} -1 & -50 \\ 10 & 6 \end{pmatrix}$$

3. The ratio of Kaitlin's age now to her father's age six years ago is 1:2. In twelve years the ratio of their ages will be 5:9. Find Kaitlin's age in years now.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1999

ROUND 3 – Geometry: Angles and Triangles

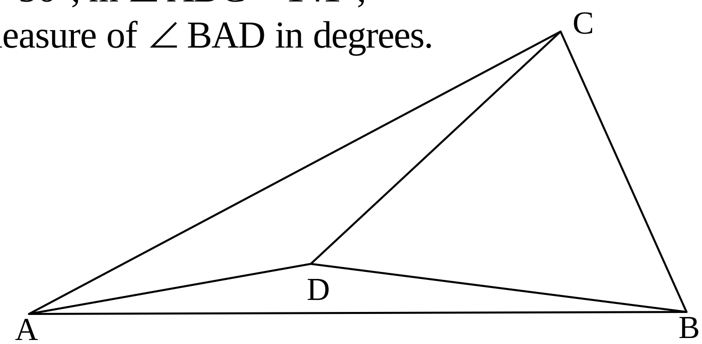
1. ____________________

2. ____________________

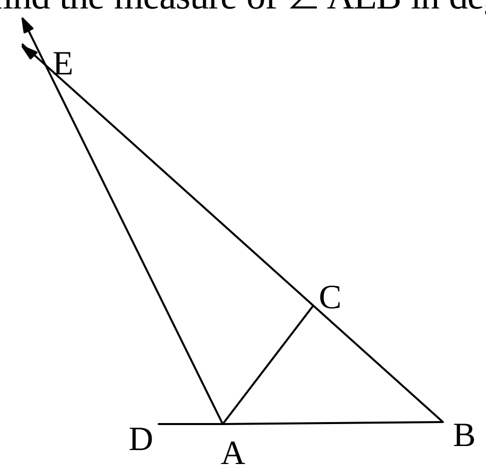
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given m $\angle$ CAB = 36°, m $\angle$ BCD = 56°, m $\angle$ ADC = 141°, AC = AB, and CB = CD, find the measure of $\angle$ BAD in degrees.



2. If m $\angle$ CAD = $\left(2x^2\right)°$, m $\angle$ ABC = $\left(5x+2\right)°$, m $\angle$ ACB = $\left(10x+6\right)°$, and the bisector of $\angle$ CAD intersects $\overrightarrow{BC}$ at point E, find the measure of $\angle$ AEB in degrees.



3. The measures of consecutive angles of a convex polygon are 171°, 173°, 176°, 171°, 173°, 176°, … , 171°, 173°, 176°, the same group of three measures appearing an integral number of times. Find the number of sides for this polygon.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1999

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all real solutions for x for the following equation: $\dfrac{2}{x+6}+\dfrac{12}{x^2+6x}=3$

2. Given the quadratic equation, $4x^2+kx+7=0$, which has two positive roots whose difference is 3, solve for k.

3. A motor boat travels 27 miles downstream helped by a current of 9 mph. A shorter water route of 21 miles is then found and the boat moves upstream against a 1 mph current. If the entire trip took 6 hours, and the boat maintains a constant speed, find this speed in miles per hour.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1999

ROUND 5 – Trig. Equations

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given $0° \le x < 360°$ and $\sin x + \tan 60° \cos x = 0$, find all solutions for *x*.

2. Given $0° \le x < 360°$ and $\cos 2x + \sin 2x = \sin 270°$, find all solutions for *x*.

3. Given $0° \le x < 360°$ and $2 + 2\cos x = \dfrac{\sin x}{1 - \cos x}$, find all solutions for *x*.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1999

TEAM ROUND

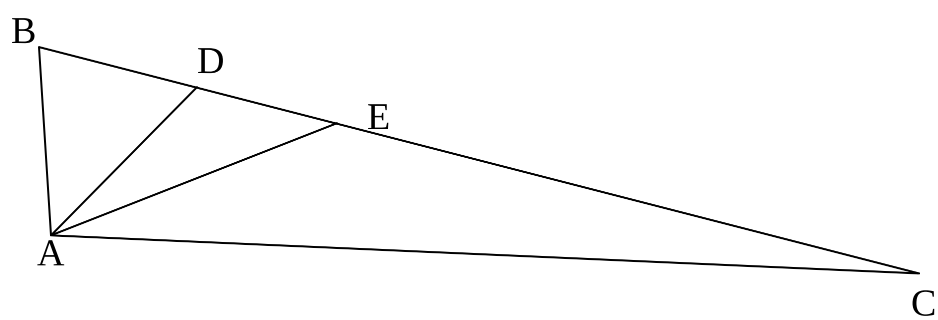
3 pts. 1. ____________________

3 pts. 2. ____________________

4 pts. 3. ____________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND

1. Given $\overline{AD}$ bisects $\angle$BAC, $\overline{AE}$ bisects $\angle$CAD, and m $\angle$ABC: m $\angle$AEB: m $\angle$ACB = 6:3:1, compute m $\angle$ADC: m $\angle$AEC.



2. If the 4 digit base 10 number $9xy1$ is divisible by 11, find the number of ordered pairs (x, y) which satisfy this condition.

3. If a 3-digit number with a non-zero units digit is subtracted from the number formed by reversing its digits, the result is between 200 and 300. The sum of its digits is 14. Find the sum of all possible 3-digit numbers satisfying these conditions.

Detailed Solutions of GBML MEET 2 – NOVEMBER 1999

ROUND 1

1. $n \cdot 55 = 25 \cdot 88 \Rightarrow n = 40$

2. Since 94z is divisible by 9, then z = 5. If z = 5, then either x or y = 5. Since 945 is not divisible by 25, then y = 5 and 945 ÷ 45 = 21. Therefore x = 1 and $x + y + z = 11$

3. $\sqrt{\sqrt{6!a}} = \sqrt[4]{6!a}$ and so 6!a must be a perfect 4th power. Since 6! = $2^4 \cdot 3^2 \cdot 5$, then $a = 3^2 \cdot 5^3 = 1125$

ROUND 2

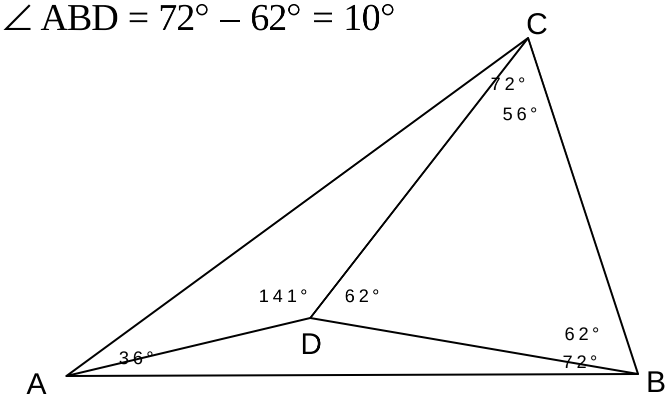
1. $\begin{cases} 3x + 4y = -a \\ 9x - 8y = 7a \end{cases} \Rightarrow 6x + 8y = -2a \Rightarrow 15x = 5a \Rightarrow x = \dfrac{a}{3} \Rightarrow y = \dfrac{-a}{2} \Rightarrow x + y = -\dfrac{a}{6}$

2. $\begin{pmatrix} x & y \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & y-4 \\ 2 & x \end{pmatrix} = \begin{pmatrix} -1 & -50 \\ 10 & 6 \end{pmatrix} \Rightarrow x + 2y = -1$ and $2y - 8 + 4x = 6 \Rightarrow 4x + 2y = 14$
$\Rightarrow 3x = 15$ and $x = 5 \Rightarrow y = -3$. Therefore $(x, y) = (5, -3)$

3. Let x = Kaitlin's age now; y = her father's age now; $\dfrac{x}{y-6} = \dfrac{1}{2}$ and $\dfrac{x+12}{y+12} = \dfrac{5}{9} \Rightarrow$
$2x = y - 6$ and $9x + 108 = 5y + 60 \Rightarrow y = 2x + 6$ and $5y - 9x = 48 \Rightarrow$
$10x + 30 - 9x = 48 \Rightarrow x = 18$

ROUND 3

1. m $\angle$ ACB = m $\angle$ ABC = 72°; m $\angle$ CDB = m $\angle$ CBD = 62°;
m $\angle$ ADB = 360° – 141° – 62° = 157°; m $\angle$ ABD = 72° – 62° = 10°
m $\angle$ BAD = 180° – 157° – 10° = 13°



2. By the exterior angle theorem:
$2x^2 = 15x + 8 \Rightarrow 2x^2 - 15x - 8 = 0 \Rightarrow (2x+1)(x-8) = 0 \Rightarrow x = 8$;
Note $x = -\frac{1}{2}$ leads to a negative measure for $\angle$ ABC.
m $\angle$ CAD = 128°; m $\angle$ ABC = 42°; Therefore m $\angle$ AEB = 64° – 42° = 22°

3. The exterior angles are 9°, 7°, 4°, 9°, 7°, 4°, …, 9°, 7°, 4° where every three add to 20°; since the exterior angles must add to 360°, the number of sides = 360 ÷ 20 × 3 = 54.

ROUND 4

1. $\frac{2}{x+6} + \frac{12}{x^2+6x} = 3 \Rightarrow 2x + 12 = 3x^2 + 18x \Rightarrow 3x^2 + 16x - 12 = 0 \Rightarrow (3x-2)(x+6) = 0$
$\Rightarrow x = \frac{2}{3}$ since x = - 6 is extraneous

2. Call the roots r and s, with $r > s$; $r - s = 3$ and $rs = \frac{7}{4} \Rightarrow r(r-3) = \frac{7}{4} \Rightarrow 4r^2 - 12r - 7 = 0$
$\Rightarrow (2r+1)(2r-7) = 0 \Rightarrow r = \frac{7}{2} \Rightarrow s = \frac{1}{2}; \frac{k}{4} = -(r+s) \Rightarrow k = -16$

3. Time down the first waterway = $\frac{27}{x+9}$; time up the second waterway = $\frac{21}{x-1}$
Equation: $\frac{27}{x+9} + \frac{21}{x-1} = 6$
$\Rightarrow 27x - 27 + 21x + 189 = 6x^2 + 48x - 54 \Rightarrow 6x^2 = 216 \Rightarrow x^2 = 36 \Rightarrow x = 6$

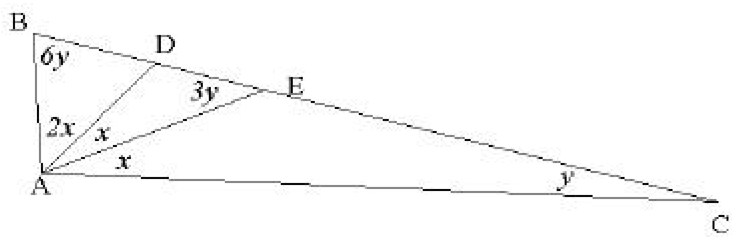
ROUND 5

1. $\sin x + \tan 60° \cos x = 0 \Rightarrow \sin x = -\sqrt{3}\cos x \Rightarrow \tan x = -\sqrt{3} \Rightarrow x = 120°, 300°$

2. $\cos 2x + \sin 2x = \sin 270° \Rightarrow 2\cos^2 x - 1 + 2\sin x \cos x = -1 \Rightarrow 2\cos x\left(\cos x + \sin x\right) = 0$
$\Rightarrow \cos x = 0$ or $\tan x = -1 \Rightarrow x = 90°, 135°, 270°, 315°$

3. $2 + 2\cos x = \dfrac{\sin x}{1 - \cos x} \Rightarrow 2\left(1 + \cos x\right)\left(1 - \cos x\right) = \sin x \Rightarrow 2\left(1 - \cos^2 x\right) = \sin x \Rightarrow$
$2\sin^2 x - \sin x = 0 \Rightarrow \sin x\left(2\sin x - 1\right) = 0 \Rightarrow \sin x = 0$ or $\sin x = \dfrac{1}{2} \Rightarrow$
$x = 0°, 180°, 30°, 150°$, but $x = 0°$ is extraneous since $1 - \cos 0° = 0 \Rightarrow x = 30°, 150°, 180°$

TEAM ROUND



1. $3y = y + x \Rightarrow x = 2y$;
$4x + 7y = 180° \Rightarrow$
$15y = 180° \Rightarrow y = 12°$;
m $\angle$ ADC =
$2x + 6y = 10y = 120°$;
m $\angle$ AEC = 180° – 3y = 144°; m $\angle$ ADC: m $\angle$ AEC = 120:144= 5:6

2. $9xy1$ is divisible by 11 $\Rightarrow (9 + y) - (x + 1) = 0, 11, 22.... \Rightarrow y = x - 8$ or $y = x + 3$
Note 22 is too large to produce any ordered pairs.
If $y = x - 8$ results in ordered pairs (8, 0) and (9,1)
If $y = x + 3$results in ordered pairs (0, 3),(1, 4) … (6, 9).
Therefore there are 9 possibilities

3. Call the original number, $100h + 10t + u$. The two results are $h + t + u = 14$ and
$200 < \left(100u + 10t + u\right) - \left(100h + 10t + u\right) < 300 \Rightarrow 200 < 99u - 99h < 300 \Rightarrow$
$200 < 99\left(u - h\right) < 300 \Rightarrow u - h = 3$; adding the equations: $2u + t = 17$ or $t = 17 - 2u$;
now list all the possibilities: $u = 4 \Rightarrow t = 9 \Rightarrow h = 1$ (194); $u = 5 \Rightarrow t = 7 \Rightarrow h = 2$ (275);
$u = 6 \Rightarrow t = 5 \Rightarrow h = 3$ (356); $u = 7 \Rightarrow t = 3 \Rightarrow h = 4$ (437); $u = 8 \Rightarrow t = 1 \Rightarrow h = 5$ (518);
Finally adding the five possibilities: $518 + 437 + 356 + 275 + 194 = 1780$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1999

ANSWER SHEET:

ROUND 1

1. 40
2. 11
3. 1125

ROUND 2

1. $-\frac{a}{6}$
2. (5, -3)
3. 18

ROUND 3

1. 13 (13°)
2. 22 (22°)
3. 54

ROUND 4

1. $\frac{2}{3}$
2. – 16
3. 6 (6 mph)

ROUND 5

1. 120°, 300°
2. 90°, 135°, 270°, 315°
3. 30°, 150°, 180°

TEAM ROUND

3 pts. 1. 5:6 $\left(\frac{5}{6}\right)$

3 pts. 2. 9

4 pts. 3. 1780

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2000

ROUND 1 – Arithmetic-Open

1. (, ,)

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given x, y, and z are distinct, non-zero digits in base 8 satisfying the following addition in base 8 with $x > y$, find the ordered triple (x, y, z).

$$\begin{array}{r} x\ z_8 \\ +\ y\ z_8 \\ \hline 1\ 4\ 0_8 \end{array}$$

2. How many natural (counting) numbers less than 199 are divisible by 3 or 5, but not by both 3 and 5?

3. How many counting (natural) numbers less than 100 have 12 positive integral factors?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2000

ROUND 2 – Simultaneous Linear Equations, Word Problems, Matrices

1. _(,)__

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the ordered pair (x, y) which is a solution to the following matrix equation.

$$\begin{pmatrix} x & 2 \\ y & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} y-10 \\ 4-x \end{pmatrix}$$

2. Find the 2-digit whole number such that three times its ten's digit is one less than seven times its unit's digit and when the number formed by reversing its digits is subtracted from the number, the result is 45.

3. One amount of money is invested at 5% and another amount is invested at 8%. The total yearly interest from both investments is $620. If the interest rates were reversed on the two amounts, the annual interest would be increased by $60. What is the total number of dollars invested?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2000

ROUND 3 – Geometry: Angles and Triangles

1. ____________________

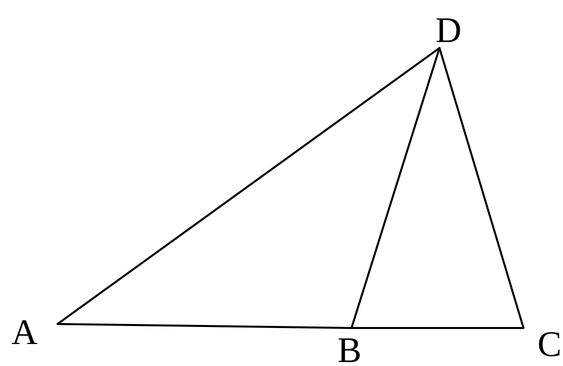
2. ____________________

3. ____________________

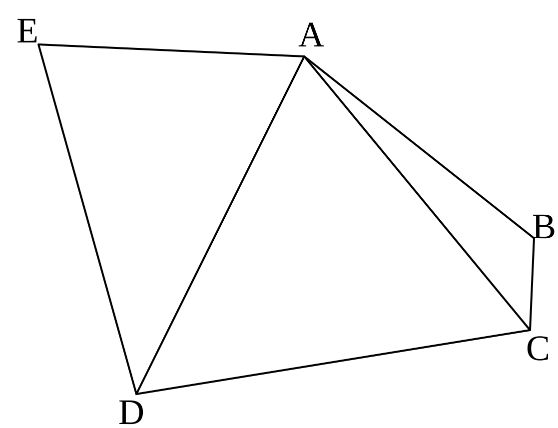
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The supplement of the complement of an angle is 6° less than the supplement of the angle. Find the number of degrees in the measurement of the angle.

2. Given $\overline{ABC}$, AB = BD = CD, and $m\angle ADC = 66°$, compute the number of degrees in $m\angle C$.



3. The ratio of the measures of consecutive exterior angles of convex pentagon ABCDE at vertices A, B, C, D, and E is 2:3:4:5:6, respectively. If $\overrightarrow{AD}$ bisects $\angle CAE$ and $\angle ADE \cong \angle ACB$, compute the number of degrees in $m\angle BAC$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2000

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

1. ____________________

2. ____________________

3. (,)

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Kaitlin can paint a house in 6 hours less time than Abbe. Kaitlin paints the house alone for 6 hours then stops. Now Abbe comes in and finishes painting the house in 16 hours. How many hours does it take Kaitlin to paint the entire house working alone?

2. The following quadratic equation in x has the property that the product of its roots is 8 more than the sum of its roots. Find all possible values for *k*.

   $kx^2 + k^2x - x + 8 = 0$

3. Given the quadratic equations, $x^2 - 3x - 5 = 0$ and $x^2 + bx + c = 0$, such that the second equation has solutions which are the squares of the solutions to the first equation, find the ordered pair (b, c).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2000

ROUND 5 – Trig. Equations

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given $0° \leq x < 360°$ and $\cos^2 x + \cos x \cdot \sin x = 0$, find all solutions for *x*.

2. Given $0° < x < 45°$ and $\sin x + \cos x = \frac{4}{3}$, compute $\cos 2x$ in simplest radical form.

3. Given $0° \leq x < 360°$ and $3\sec^4 x - 3\tan^4 x = 5$, find all solutions for *x*.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2000

TEAM ROUND

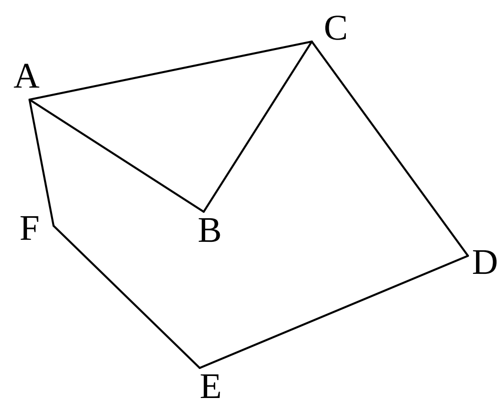
3 pts. 1. ____________________

3 pts. 2. ____________________

4 pts. 3. ____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given the figure on the right such that $m\angle E - m\angle ABC = 24°$, $m\angle F - m\angle D = 70°$, and $m\angle BCD - m\angle BAF = 22°$, compute the number of degrees in $m\angle BCD + m\angle D$.



2. If the equation $\frac{x}{x+1}+\frac{x+2}{x-1}=k$ has only one solution for x, find exactly all possible values for k.

3. An inheritance between $50,000 and $51,000 when divided almost evenly among seven heirs, two get an extra dollar, when divided almost evenly among eleven heirs, three get an extra dollar, and when divided among thirteen heirs, they all get the same amount. Find the number of dollars in this inheritance.

Detailed Solutions of GBML MEET 2 – NOVEMBER 2000

ROUND 1

1. Since $z \neq 0,\ z+z=8 \rightarrow z=4 \rightarrow x+\ y+\ 1=14_8 \rightarrow x+y=11 \rightarrow x=6,\ y=5$. [Note since the digits are distinct $\rightarrow\ x=7,\ y=4$ is not possible.] The triple is $(6,5,4)$.

2. Find how many multiples of 3, 5, and 15 are less than 199:
$199 \div 3 = 66\tfrac{1}{3}$; $199 \div 5 = 39\tfrac{4}{5}$; $199 \div 15 = 13\tfrac{4}{15}$; therefore the result = 66 + 39 – 13 – 13 = 79.

3. Since the number has 12 factors it is of four types: $(i)\,p^{11}\ \ (ii)\,p^5q\ \ (iii)\,p^3q^2\ \ (iv)\,p^2qr$, where *p*, *q*, and *r* are prime. Since the number is less than 100 $\rightarrow$ none of type (i);
$2^5 \cdot 3$ of type (ii); $2^3 \cdot 3^2$ of type (iii); $2^2 \cdot 3 \cdot 5,\ 2^2 \cdot 3 \cdot 7,\ 3^2 \cdot 2 \cdot 5$ of type (iv); therefore there are 5 numbers less than 100 with 12 factors.

ROUND 2

1. $\begin{pmatrix} x & 2 \\ y & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} y-10 \\ 4-x \end{pmatrix} \rightarrow \begin{cases} 4x+14=y-10 \\ 4y-7=4-x \end{cases} \rightarrow \begin{cases} 4x+24=y \\ 4y+x=11 \end{cases} \rightarrow$
$16x+96+x=11 \rightarrow 17x=-85 \rightarrow x=-5,\ y=4 \rightarrow$ ordered pair solution is $(-5,4)$.

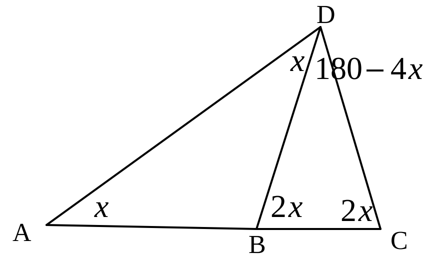
2 Let *t* = ten's digit, *u* = unit's digit; $\rightarrow \begin{cases} 3t=7u-1 \\ (10t+u)-(10u+t)=45 \end{cases} \rightarrow \begin{cases} -7u+3t=-1 \\ 9t-9u=45 \end{cases} \rightarrow$
$\begin{cases} -7u+3t=-1 \\ 3u-3t=-15 \end{cases} \rightarrow -4u=-16 \rightarrow u=4 \rightarrow t=9 \rightarrow$ number is 94

3. $\begin{cases} .05x+.08y=620 \\ .08x+.05y=680 \end{cases} \rightarrow \begin{cases} .40x+.64y=4960 \\ -.40x-.25y=-3400 \end{cases} \rightarrow .39y=1560 \rightarrow y=4000 \rightarrow$
$.05x+320=620 \rightarrow .05x=300 \rightarrow x=6000 \rightarrow x+y=10000$

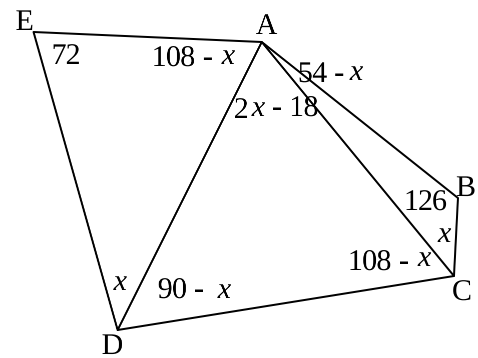
ROUND 3

1. $180-(90-x)=(180-x)-6 \rightarrow 90+x=174-x \rightarrow 2x=84 \rightarrow x=42$

2. $180-4x+x=66 \rightarrow 3x=114 \rightarrow x=38$
$\rightarrow 2x=76$



3. $2a+3a+4a+5a+6a=360 \rightarrow 20a=360 \rightarrow$
$a=18 \rightarrow m\angle ABC=180-3(18)=126$;
$m\angle BCD=180-4(18)=108$;
$m\angle CDE=180-5(18)=90$
$m\angle AED=180-6(18)=72$; $m\angle ADC=90-x$ and
$m\angle ACD=108-x \rightarrow m\angle DAC=2x-18$;
$m\angle DAE=108-x \rightarrow 108-x=2x-18 \rightarrow x=42 \rightarrow$
$m\angle BAC=54-42=12$



ROUND 4

1. Let x = number of hours Kaitlin takes to paint the house $\rightarrow x+6$ =number of hours Abbe takes to paint the house; $\frac{6}{x}+\frac{16}{x+6}=1 \rightarrow 6x+36+16x=x^2+6x \rightarrow$
$x^2-16x-36=0 \rightarrow (x-18)(x+2)=0 \rightarrow x=18$

2. $kx^2+k^2x-x+8=0 \rightarrow kx^2+(k^2-1)x+8=0 \rightarrow$ product of its roots = $\frac{8}{k}$ and the sum of its roots = $-\frac{k^2-1}{k}=\frac{1-k^2}{k} \rightarrow \frac{8}{k}=\frac{1-k^2}{k}+8 \rightarrow 8=1-k^2+8k \rightarrow k^2-8k+7=0 \rightarrow$
$(k-1)(k-7)=0 \rightarrow k=1,7$

3. Call the roots of the first equation r and $s \rightarrow r^2$ and s^2 are the roots of the 2nd equation
$\rightarrow r+s=3,\ rs=-5,\ r^2+s^2=-b,\ r^2s^2=c$; $r^2+s^2=(r+s)^2-2rs=3^2-2(-5)=19 \rightarrow$
and $r^2s^2=(rs)^2=(-5)^2=25 \rightarrow (b,c)=(-19,25)$

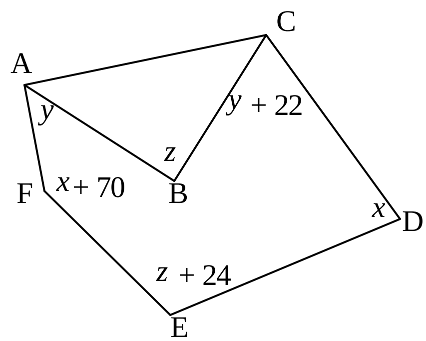
ROUND 5

1. $\cos^2 x + \cos x \cdot \sin x = 0 \rightarrow \cos x(\cos x + \sin x) = 0 \rightarrow \cos x = 0 \text{ or } \sin x = -\cos x \rightarrow$
$\cos x = 0$ or $\tan x = -1 \rightarrow x = 90°, 135°, 270°, 315°$

2. $\sin x + \cos x = \frac{4}{3} \rightarrow (\sin x + \cos x)^2 = \frac{16}{9} \rightarrow \sin^2 x + 2\sin x \cos x + \cos^2 x = \frac{16}{9} \rightarrow$
$1 + \sin 2x = \frac{16}{9} \rightarrow \sin 2x = \frac{7}{9} \rightarrow$ since $0° < 2x < 90°, \cos 2x = \sqrt{1 - \frac{49}{81}} = \sqrt{\frac{32}{81}} = \frac{4\sqrt{2}}{9}$

3. $3\sec^4 x - 3\tan^4 x = 5 \rightarrow 3(\sec^4 x - \tan^4 x) = 5 \rightarrow 3(\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x) = 5$
$\rightarrow 3(1 + 2\tan^2 x) = 5 \rightarrow 3 + 6\tan^2 x = 5 \rightarrow \tan^2 x = \frac{1}{3} \rightarrow \tan x = \pm\frac{1}{\sqrt{3}} \rightarrow$
$x = 30°, 150°, 210°, 330°$

TEAM ROUND



1. If you draw $\overline{BE}$ you see that the following is true:
$m\angle BAF + (360 - m\angle ABC) + m\angle BCD +$
$m\angle D + m\angle E + m\angle F = 720$
$\rightarrow y + 360 - z + y + 22 + x + z + 24 + x + 70 = 720 \rightarrow$
$2x + 2y = 244 \rightarrow x + y + 22 = 144$

2. $\frac{x}{x+1} + \frac{x+2}{x-1} = k \rightarrow$
$x^2 - x + x^2 + 3x + 2 = k(x^2 - 1) \rightarrow 2x^2 + 2x + 2 = kx^2 - k \rightarrow$
$(k-2)x^2 - 2x + (-k-2) = 0$; 1 solution $\rightarrow$ discriminant = 0 $\rightarrow$
$(-2)^2 - 4(k-2)(-k-2) = 0 \rightarrow 4 + 4(k-2)(k+2) = 0 \rightarrow$
$1 + k^2 - 4 = 0 \rightarrow k^2 = 3 \rightarrow k = \pm\sqrt{3}$ or if $k = 2$, then the equation becomes linear and thus has one solution. Therefore the values for k are $2, \pm\sqrt{3}$.

3. The number = $7x + 2 = 11y + 3 = 13z$; first find the smallest number such that x and y satisfies the first equation: $y = \frac{7x-1}{11} \rightarrow x = 8$ and $y = 5 \rightarrow$ number is 58 ≡ 6 mod 13; adding $77n$ to 58 produced numbers with the same property; $77 \equiv -1 \bmod 13 \rightarrow$
$58 + 6 \cdot 77 = 520 \equiv 6 + 6(-1) \bmod 13 \equiv 0 \bmod 13$; $7 \cdot 11 \cdot 13 = 1001$; adding $1001n$ to 520 produced numbers with the same property; $1001 \cdot 50 + 520 = 50570$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2000

ANSWER SHEET:

ROUND 1

1. $(6,5,4)$
2. 79
3. 5

ROUND 2

1. $(-5,4)$
2. 94
3. 10000 ($10000)

ROUND 3

1. 42 (42°)
2. 76 (76°)
3. 12 (12°)

ROUND 4

1. 18 (18 hours)
2. 1,7
3. $(-19,25)$

ROUND 5

1. $90°, 135°, 270°, 315°$
2. $\frac{4\sqrt{2}}{9}$
3. $30°, 150°, 210°, 330°$

TEAM ROUND

3 pts. 1. 144 (144°)

3 pts. 2. $\pm\sqrt{3}$, 2

4 pts. 3. 50570 ($50570)

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2001

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the smallest whole number which has a remainder of 3 when divided by 7 and has a remainder of 8 when divided by 13.

2. Let *M* equal the smallest positive multiple of five which is one less than a perfect cube. Let *N* equal the largest positive integer which is less than one thousand with exactly three factors. Find the sum of *M* and *N*.

3. Given $X4Y_{(9)} = Y4X_{(10)}$, find all possible ordered pairs (X,Y).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2001

ROUND 2 – Simultaneous Linear Equations, Word Problems, Matrices

1. __$________________

2. $\left(\_\_\_, \_\_\_ \right)$

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. In *Mathland's* monetary system, 3 *alphas* and 5 *betas* are worth $4.34 while 5 *alphas* and 3 *betas* are worth $4.30. Find the total value of 1 *alpha* and 1 *beta* in dollars and cents.

2. Given the following matrix multiplication, find the ordered pair (A, B).

$$\begin{pmatrix} 5 & A \\ 3 & 2B \end{pmatrix}\begin{pmatrix} B & 3A \\ 2 & -1 \end{pmatrix}=\begin{pmatrix} -19 & C \\ D & -12 \end{pmatrix}$$

3. Al and his children Bill and Carol have ages that total 78 years. In 10 years Al will be twice Bill's age then and 6 years ago Al was five times Carol's age then. How many years old is Al now?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2001

ROUND 3 – Geometry: Angles and Triangles

1. ____________________

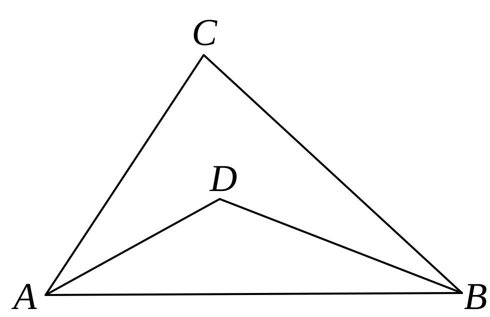
2. ____________________

3. ____________________

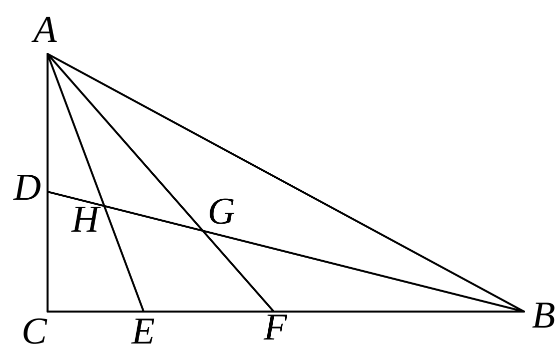
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given ΔABC with $\overline{ADC}$, $\overline{BD} \perp \overline{AC}$, and $BA : BC : BD = 2 : \sqrt{2} : 1$. Find the number of degrees in the measure of $\angle ABC$.

2. Given $\overline{AD}$ bisects $\angle BAC$, $\overline{BD}$ bisects $\angle ABC$, and $m\angle ACB + m\angle ADB = 210°$, find the number of degrees in $m\angle ACB$.



3. Given $\angle C$ is right, $\overline{BGHD}$, $\overline{AHE}$, $\overline{AGF}$, $\overline{ADC}$, $\overline{CEFB}$, $\overline{BD}$ bisects $\angle ABC$, $\overline{AE}$ and $\overline{AF}$ trisect $\angle BAC$. If $m\angle EHG - m\angle GFE = 10°$, find the number of degrees in $m\angle AGB$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2001

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

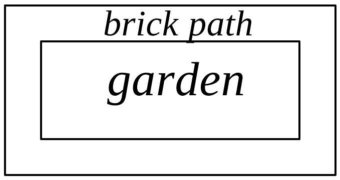
1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Mr. Green buys fencing to enclose a garden 4 times longer than it is wide. After fencing in the garden, he builds a brick path 2 feet wide around the garden. (See the figure below.) If the total area of the garden and brick path is 280 square feet, find how many feet of fencing Mr. Green bought.



2. Solve the following equation for x:

$$\frac{x}{3x-6}-\frac{2}{2x+10}=\frac{7}{x^2+3x-10}$$

3. Given the quadratic equation $x^2+bx+c=0$ has real roots whose difference is 3. If $\frac{c}{b}=-\frac{20}{3}$, find all possible values for the smaller of the two roots.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2001

ROUND 5 – Trig. Equations

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. If $\tan x = 5$, find the possible value(s) of $\dfrac{2\sin x + 3\cos x}{5\sin x + \cos x}$.

2. Given $0° \le x < 360°$ and $\tan x + \sec x = \cos x$, find all solutions for x.

3. Given $0° \le x < 360°$ and $\tan^3 x + \sec^2 x = 3\tan x + 4$, find the sum of all solutions for x.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2001

TEAM ROUND (<u>12 MINUTES LONG</u>)

3 pts. 1. ___________________

3 pts. 2. $y =$ ___________

4 pts. 3. ___________________

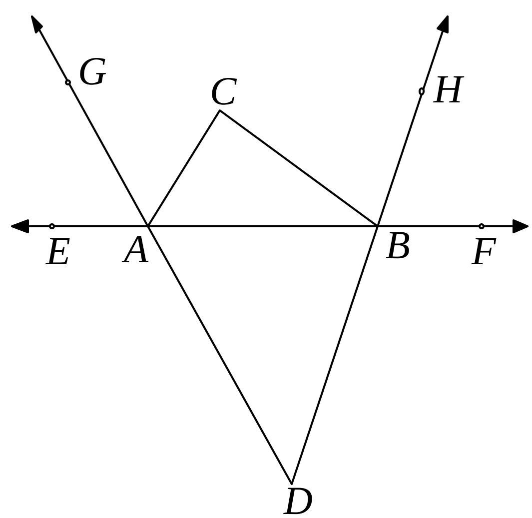
SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given the following matrix multiplication, find all possible ordered pairs (a, b).

   Write all answers as ordered pairs.

   $$\begin{pmatrix} x & y \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -5 & y \\ x & 2 \end{pmatrix} = \begin{pmatrix} 12 & -1 \\ a & b \end{pmatrix}$$

2. Given $\overline{EABF}$, $\overline{DAG}$, $\overline{DBH}$, $\overrightarrow{AG}$ bisects $\angle CAE$ and $\overrightarrow{BH}$ bisects $\angle CBF$. If $m\angle C = (x + y)^\circ$ and $m\angle D = (x - y)^\circ$, find y in terms of x.



3. The number 10! (10 factorial) has how many perfect square factors?

Detailed Solutions of GBML MEET 2 – NOVEMBER 2001

ROUND 1

1. The number is of the forms $7y+3$ or $13x+8 \Rightarrow y=\dfrac{13x+5}{7} \Rightarrow$ when $x=5$, $y=10 \Rightarrow$ the number = $7\times 10+3=73$.

2. $M=6^3-1=215$; N must be the square of a prime to have exactly 3 factors $\Rightarrow$ $N=31^2=961 \Rightarrow M+N=215+961=1176$.

3. $X4Y_{(9)}=Y4X_{(10)} \Rightarrow 81X+36+Y=100Y+40+X \Rightarrow 80X=99Y+4 \Rightarrow$ when $Y=4$ then $80X=99\cdot 4+4=400 \Rightarrow X=5 \Rightarrow (X,Y)=(5,4)$.

ROUND 2

1. $3a+5b=4.34$ and $5a+3b=4.30 \Rightarrow 8a+8b=8.64 \Rightarrow a+b=\$1.08$

2 $\begin{cases} 2A+5B=-19 \\ 9A-2B=-12 \end{cases} \Rightarrow \begin{cases} 4A+10B=-38 \\ 45A-10B=-60 \end{cases} \Rightarrow 49A=-98 \Rightarrow$
$A=-2 \Rightarrow -4+5B=-19 \Rightarrow B=-3 \Rightarrow (A,B)=(-2,-3)$.

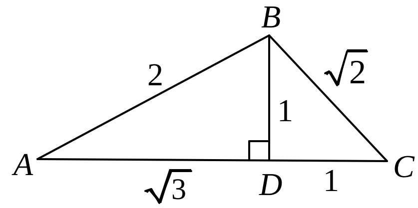
3. Let a = Al's current age; b = Bill's current age; c = Carol's current age $\Rightarrow$

$$\begin{cases} a+b+c=78 \\ a+10=2(b+10) \\ a-6=5(c-6) \end{cases} \Rightarrow \begin{cases} a+b+c=78 \\ a=2b+10 \\ a=5c-24 \end{cases} \Rightarrow \begin{cases} a+b+c=78 \\ b=\dfrac{a-10}{2} \\ c=\dfrac{a+24}{5} \end{cases} \Rightarrow a+\frac{a-10}{2}+\frac{a+24}{5}=78 \Rightarrow$$

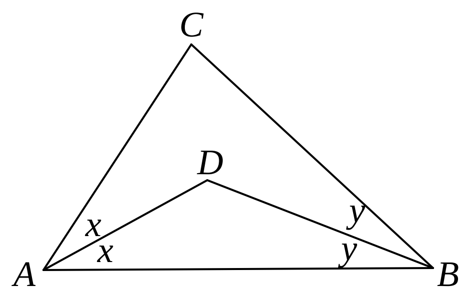
$10a+5a-50+2a+48=780 \Rightarrow 17a=782 \Rightarrow a=46$.

ROUND 3

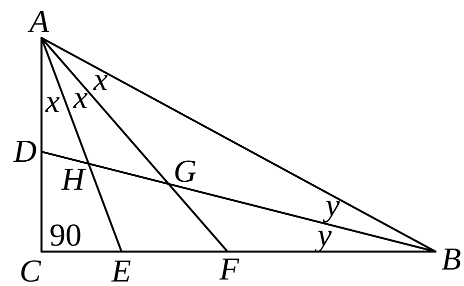
1. There is no loss of generality to let $AB = 2$, $BD = 1$ and $BC = \sqrt{2} \Rightarrow AD = \sqrt{3}$ and $DC = 1 \Rightarrow$ $m\angle ABD = 60°$ *and* $m\angle CBD = 45° \Rightarrow m\angle ABC = 105°$.



2. Let $z = x + y \Rightarrow m\angle ACB = (180 - 2z)°$ and $m\angle ADB = (180 - z)° \Rightarrow 360 - 3z = 210 \Rightarrow z = 50 \Rightarrow$ $m\angle ACB = 80°$.



3. Let $m\angle CAE = m\angle FAE = m\angle BAF = x°$; let $m\angle ABD = m\angle CBD = y°$; $3x + 2y = 90$; by the exterior angle theorem, $m\angle EHG = (2x + y)°$ and $m\angle GFE = (2y + x)° \Rightarrow (2x + y) - (2y - x) = 10$ $\Rightarrow x - y = 10 \Rightarrow 2x - 2y = 20 \Rightarrow 5x = 110 \Rightarrow x = 22 \Rightarrow$ $y = 12 \Rightarrow m\angle AGB = (180 - 22 - 12)° = 146°$.



ROUND 4

1. Let x = width of garden $\Rightarrow 4x$ = length of garden $\Rightarrow$ length of fence = $10x$; $(x+4)(4x+4) = 280 \Rightarrow (x+4)(x+1) = 70 \Rightarrow x^2 + 5x + 4 = 70 \Rightarrow x^2 + 5x - 66 = 0 \Rightarrow$ $(x+11)(x-6) = 0 \Rightarrow x = 6 \Rightarrow$ fencing = 60 ft.

2. $\dfrac{x}{3x-6} - \dfrac{2}{2x+10} = \dfrac{7}{x^2+3x-10} \Rightarrow \dfrac{x}{3(x-2)} - \dfrac{1}{x+5} = \dfrac{7}{(x-2)(x+5)} \Rightarrow$ $x(x+5) - 3(x-2) = 21 \Rightarrow x^2 + 5x - 3x + 6 = 21 \Rightarrow x^2 + 2x - 15 = 0 \Rightarrow$ $(x+5)(x-3) = 0 \Rightarrow x = 3$ (since $x = -5$ is extraneous to the equation.)

3. Call the smaller root of the equation $r \Rightarrow$ larger root = $r + 3$; $b = -(2r+3)$ and $c = r(r+3) \Rightarrow \dfrac{r(r+3)}{-(2r+3)} = -\dfrac{20}{3} \Rightarrow 3r^2 + 9r = 40r + 60 \Rightarrow$ $3r^2 - 31r - 60 = 0 \Rightarrow (3r+5)(r-12) = 0 \Rightarrow r = -\dfrac{5}{3}, 12$

ROUND 5

1. $\dfrac{2\sin x+3\cos x}{5\sin x+\cos x} = \dfrac{2\tan x+3}{5\tan x+1}=\dfrac{2(5)+3}{5(5)+1}=\dfrac{13}{26}=\dfrac{1}{2}$

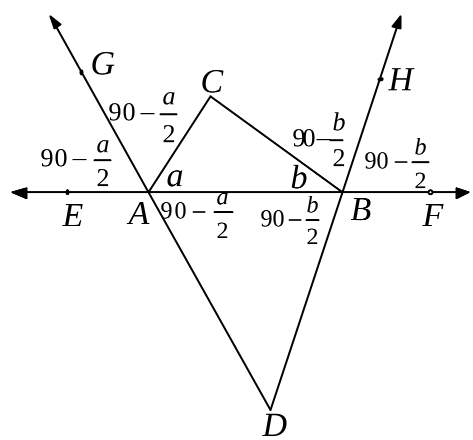
2. $\tan x+\sec x=\cos x \Rightarrow \dfrac{\sin x}{\cos x}+\dfrac{1}{\cos x}=\cos x \Rightarrow \sin x+1=\cos^2 x,(\text{and } \cos x\neq 0)\Rightarrow$
$\sin x+1=1-\sin^2 x \Rightarrow \sin^2 x+\sin x=0 \Rightarrow \sin x(\sin x+1)=0 \Rightarrow \sin x=0,-1 \Rightarrow x=0°,180°$
since $\sin x=-1 \Rightarrow x=270° \Rightarrow \cos x=0$

3. $\tan^3 x+\sec^2 x=3\tan x+4 \Rightarrow \tan^3 x+\tan^2 x+1=3\tan x+4 \Rightarrow$
$\tan^3 x+\tan^2 x-3\tan x-3=0 \Rightarrow \tan^2 x(\tan x+1)-3(\tan x+1)=0 \Rightarrow$
$(\tan^2 x-3)(\tan x+1)=0 \Rightarrow \tan x=\pm\sqrt{3},\ -1 \Rightarrow x=60°,120°,240°,300°,135°,315°;$
the sum of all values for x = 1170°.

TEAM ROUND

1. $\begin{pmatrix} x & y \\ -2 & 4\end{pmatrix}\begin{pmatrix} -5 & y \\ x & 2\end{pmatrix}=\begin{pmatrix} 12 & -1 \\ a & b\end{pmatrix} \Rightarrow \begin{cases} -5x+xy=12 \\ xy+2y=-1\end{cases} \Rightarrow \begin{cases} y=\dfrac{12+5x}{x} \\ y=\dfrac{-1}{x+2}\end{cases} \Rightarrow \dfrac{-1}{x+2}=\dfrac{12+5x}{x} \Rightarrow$
$-x=5x^2+22x+24 \Rightarrow 5x^2+23x+24=0 \Rightarrow (5x+8)(x+3)=0 \Rightarrow$
$x=-1.6 \Rightarrow y=-2.5,\ x=-3 \Rightarrow y=1;\ \begin{pmatrix} -1.6 & -2.5 \\ -2 & 4\end{pmatrix}\begin{pmatrix} -5 & -2.5 \\ -1.6 & 2\end{pmatrix}=\begin{pmatrix} 12 & -1 \\ 3.6 & 13\end{pmatrix}$
$\begin{pmatrix} -3 & 1 \\ -2 & 4\end{pmatrix}\begin{pmatrix} -5 & 1 \\ -3 & 2\end{pmatrix}=\begin{pmatrix} 12 & -1 \\ -2 & 6\end{pmatrix} \Rightarrow (a,b)=(3.6,13),(-2,6)$

2. Let $m\angle CAB=a°$ and $m\angle CBA=b° \Rightarrow$
$m\angle C=(180-(a+b))°;$
$m\angle D=\left(180-\left(90-\dfrac{a}{2}\right)-\left(90-\dfrac{b}{2}\right)\right)°=\left(\dfrac{a+b}{2}\right)°;$
$x+y=180-(a+b)$ *and* $2x-2y=a+b \Rightarrow 3x-y=180$
$\Rightarrow y=3x-180.$



3. $10!=2^8\cdot 3^4\cdot 5^2\cdot 7$; any perfect square factor contains
0 or 2 or 4 or 6 or 8 factors of 2 (5 possibilities);
any perfect square factor contains 0 or 2 or 4 factors of 3 (3 possibilities);
any perfect square factor contains 0 or 2 factors of 5 (2 possibilities);
by the basic counting principle the number of perfect square factors of 10! =
5·3·2 = 30 possibilities.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2001

ANSWER SHEET:

ROUND 1

1. 73
2. 1176
3. $(5,4)$

ROUND 2

1. \$1.08
2. $(-2,-3)$
3. 46

ROUND 3

1. 105 (105°)
2. 80 (80°)
3. 146 (146°)

ROUND 4

1. 60 (60 ft.)
2. 3
3. $-\frac{5}{3}, 12$

ROUND 5

1. $\frac{1}{2}$ (0.5)
2. 0°, 180°
3. 1170°

TEAM ROUND

3 pts. 1. $(-2,6),(3.6,13)$

3 pts. 2. $y = 3x - 180$

4 pts. 3. 30

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The number m is the smallest positive multiple of 17 such that 3 more than m is a multiple of 7. Find the value of m.

2. The $\Diamond$ operation on pairs of numbers is defined as follows: $a \Diamond b = \dfrac{ab}{a+b}$. Find all possible values of a such that a and $a \Diamond 3$ are both whole numbers.

3. Given $75 \times 196 \times 567 = 18^a \times 21^b \times 35^c$, where a, b, and c are positive integers, find the value of $a^2 + b^2 + c^2$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ROUND 2 – Simultaneous Linear Equations, Word Problems, Matrices

1. $(\quad , \quad)$

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the ordered pair (n, p) which is the solution to the following system of equations.

$$\begin{cases} 5(n-2)-7=8p \\ 2(p-4)+3n=14-n \end{cases}$$

2. On the first day of a trip, Albert spent 20% of his money and Sophia spent 30% of hers, leaving them with a total of \$820. Had, instead, Albert spent 40% of his money and Sophia spent 65% of hers, Albert would then have \$90 more to spend than Sophia. What was the total number of dollars that Albert and Sophia started with at the beginning of their trip?

3. Given the following matrix equation, $\begin{bmatrix} x & 3x \\ 2x & z \end{bmatrix} + y\begin{bmatrix} 2 & w \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & z \\ -5 & w-2 \end{bmatrix}$,

find the sum $w+z$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ROUND 3 – Geometry: Angles and Triangles

1. ____________________

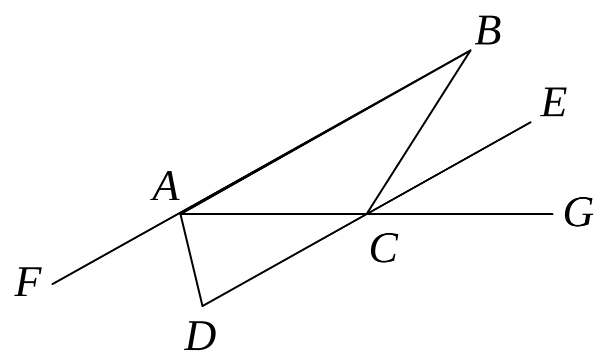
2. ____________________

3. ____________________

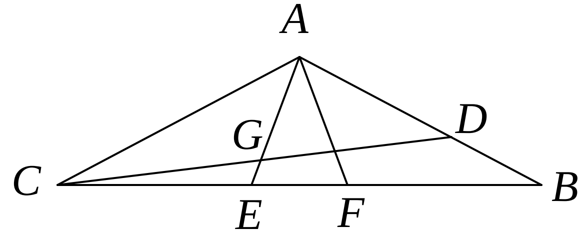
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given ΔABC with the measures of $\angle A$, an exterior angle at B, and an exterior angle at C in the ratio of 1:4:5 respectively, find the number of degrees in the measure of $\angle A$.

2. Given the diagram on the right in which $\overline{BF} \parallel \overline{DE}$, $\overrightarrow{CE}$ bisects $\angle BCG$, $\overrightarrow{AD}$ bisects $\angle GAF$, and $m\angle B + m\angle D = 112°$, find the number of degrees in $m\angle B$.



3. Given $AC = AB$, $\overline{AE}$ and $\overline{AF}$ trisect $\angle BAC$, $m\angle ACD : m\angle BCD = 3:1$, and $m\angle ADC = 45°$, find the number of degrees in $m\angle DGE$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The ratio of rows to seat per row in an auditorium was 5:2. After renovations, 10 rows were added, 2 seats were added to every row, and the auditorium now has 1820 seats. How many seats were in the auditorium before renovations?

2. The quadratic equation in x, $x^2+bx+c=0$, has roots $-3\pm 3\sqrt{11}$. Find the roots to the equation $x^2+bx+c=-18$.

3. A boat travels a certain distance upstream and the same distance downstream. If the stream's current is $4\sqrt{3}$ miles per hour and boat averages 13 miles per hour for the entire trip upstream and downstream, find the number of miles per hour in the boat's speed without a current.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ROUND 5 – Trig. Equations

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given $0° \le x < 360°$, $\cos x > 0$, and $3\tan x = \cot x$, find all solutions for *x*.

2. Given $0° \le x < 360°$ and $\sin 2x + 2\cos x - \cos^2 x = \sin x + \sin^2 x$, find all solutions for *x*.

3. Given $0° \le x < 360°$ and $\sin x = \frac{1}{2}\sqrt{8\cos x + 7}$, find all solutions for *x*.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

TEAM ROUND

3 pts. 1. ___________________

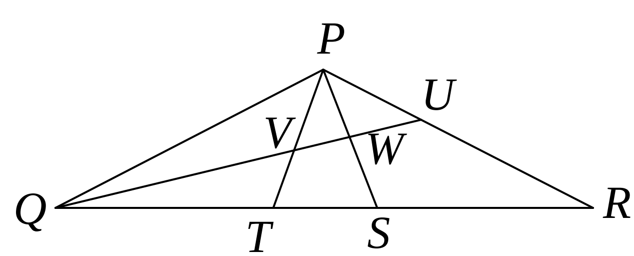
3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given n is a composite (non-prime) whole number and $72n$ has exactly 24 whole number factors, find all possible values for n.

2. Given $PQ = PR$, $\overrightarrow{PT}$ and $\overrightarrow{PS}$ trisect $\angle QPR$, $\overrightarrow{QU}$ bisects $\angle PQR$, and $m\angle QWS = 105°$, find the number of degrees in $m\angle UVT$.



3. Al has $7.85 in nickels, dimes and quarters. There are 9 more quarters than nickels. If Al has at least one of each type of coin, what is the difference between the most number of coins and least number of coins Al can have?

Detailed Solutions of GBML MEET 2 – NOVEMBER 2002

ROUND 1

1. $17+3=20,\ 34+3=37,\ 51+3=54,\ 68+3=71,\ 85+3=88, 102+3=105$, which is a multiple of 7. Answer is 102.
Alternative solution: $20\equiv -1 \bmod 7, 17\equiv 3 \bmod 7,\ -1+5(3)\equiv 0 \bmod 7 \Rightarrow 20+5(17)=105$ is the multiple of 7 $\Rightarrow$ 102 is the multiple of 17.

2. $a\lozenge 3=\dfrac{3a}{a+3}=3-\dfrac{9}{a+3}$; since a is a whole number, the only possibilities are 0 and 6.

3. $75\times 196\times 567=18^a\times 21^b\times 35^c \Rightarrow 3\times 5^2\times 2^2\times 7^2\times 7\times 3^4=\left(2\cdot 3^2\right)^a(3\cdot 7)^b(5\cdot 7)^c \Rightarrow$
$2^2\times 3^5\times 5^2\times 7^3=2^a\times 3^{2a+b}\times 5^c\times 7^{b+c} \Rightarrow a=2,\ c=2, \text{and } b=1 \Rightarrow a^2+b^2+c^2=9.$

ROUND 2

1. $\begin{cases} 5(n-2)-7=8p \\ 2(p-4)+3n=14-n \end{cases} \Rightarrow \begin{cases} 5n-17=8p \\ 2p-8+3n=14-n \end{cases} \Rightarrow \begin{cases} 5n-8p=17 \\ 4n+2p=22 \end{cases} \Rightarrow \begin{cases} 5n-8p=17 \\ 16n+8p=88 \end{cases} \Rightarrow$
$21n=105 \Rightarrow n=5 \Rightarrow 20+2p=22 \Rightarrow p=1 \Rightarrow$ solution is $(5,1)$.

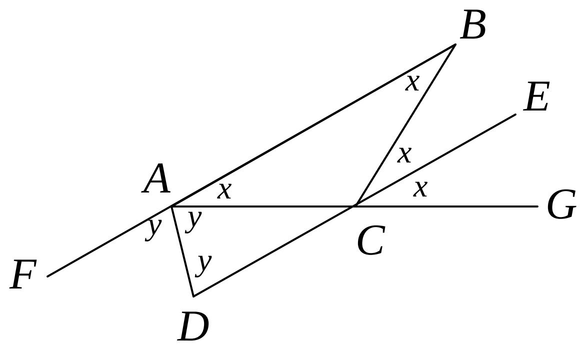
2 Let x = Albert's money, y = Sophia's money $\Rightarrow$
$\begin{cases} 0.8x+0.7y=820 \\ 0.6x-0.35y=90 \end{cases} \Rightarrow \begin{cases} 0.8x+0.7y=820 \\ 1.2x-0.7y=180 \end{cases} \Rightarrow$
$2x=1000 \Rightarrow x=500 \Rightarrow 400+.7y=820 \Rightarrow .7y=420 \Rightarrow y=600 \Rightarrow x+y=1100$

3. $\begin{cases} x+2y=-4 \\ 2x+3y=-5 \end{cases} \Rightarrow \begin{cases} -2x-4y=8 \\ 2x+3y=-5 \end{cases} \Rightarrow -y=3 \Rightarrow y=-3 \Rightarrow x-6=-4 \Rightarrow x=2 \Rightarrow$
$\begin{cases} 6-3w=z \\ z-12=w-2 \end{cases} \Rightarrow 6-3w-12=w-2 \Rightarrow 4w=-4 \Rightarrow w=-1 \Rightarrow z=9 \Rightarrow w+z=8$

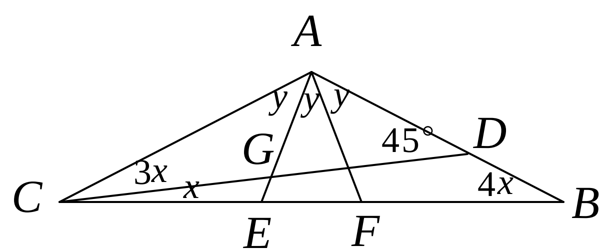
ROUND 3

1. Let $x = m\angle A \Rightarrow 4x =$ measure of exterior angle at B and $5x =$ measure of exterior angle at $C \Rightarrow 5x = x + 180 - 4x \Rightarrow 8x = 180 \Rightarrow x = 22.5$

2. Let $m\angle BCE = m\angle GCE = x \Rightarrow m\angle BAG = m\angle ABC = x$ by corresponding and alternate interior angles, respectively.
Let $m\angle GAD = m\angle FAD = y \Rightarrow m\angle ADE = y$ by alternate interior angles. $x + y = 112$ and $x + 2y = 180 \Rightarrow y = 68 \Rightarrow x = 44$.



3. Let $m\angle BCD = x \Rightarrow m\angle ACD = 3x \Rightarrow m\angle B = 4x$; let $m\angle CAE = m\angle EAF = m\angle BAF = y$; $5x = 45 \Rightarrow x = 9 \Rightarrow 4x = 36 \Rightarrow 3y + 72 = 180 \Rightarrow y = 36$; $m\angle DGE = m\angle AGC = 2y + 45 = 72 + 45 = 117$.



ROUND 4

1. Let $5x$ = number of rows $\Rightarrow 2x$ = seats per row: $(5x+10)(2x+2) = 1820 \Rightarrow (x+2)(x+1) = 182 \Rightarrow x^2 + 3x - 180 = 0 \Rightarrow (x+15)(x-12) = 0 \Rightarrow x = 12 \Rightarrow$ the auditorium original number of seats was $10x^2 = 1440$.

2. sum of the roots = $-b = -6 \Rightarrow b = 6$; product of the roots = $c = 9 - 99 = -90$; new equation is: $x^2 + 6x - 90 = -18 \Rightarrow x^2 + 6x + 9 = 81 \Rightarrow (x+3)^2 = 9^2 \Rightarrow x + 3 = \pm 9 \Rightarrow x = -3 \pm 9 = -12, 6$.

3. Let r = rate of the boat without any current and d = distance one way $\Rightarrow$
$$\frac{d}{r+4\sqrt{3}} + \frac{d}{r-4\sqrt{3}} = \frac{2d}{13} \Rightarrow \frac{1}{r+4\sqrt{3}} + \frac{1}{r-4\sqrt{3}} = \frac{2}{13} \Rightarrow \frac{2r}{r^2-48} = \frac{2}{13} \Rightarrow \frac{r}{r^2-48} = \frac{1}{13} \Rightarrow$$
$13r = r^2 - 48 \Rightarrow r^2 - 13r - 48 = 0 \Rightarrow (r-16)(r+3) = 0 \Rightarrow r = 16$ miles per hour.

ROUND 5

1. $3\tan x = \cot x \Rightarrow \tan^2 x = \frac{1}{3} \Rightarrow \tan x = \pm\frac{1}{\sqrt{3}}$; since $\cos x > 0 \Rightarrow x$ is in quadrants I or IV $\Rightarrow$ $x = 30°, 330°$.

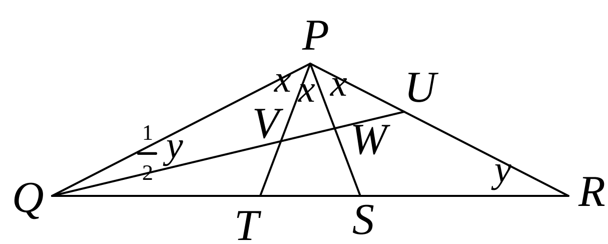
2. $\sin 2x + 2\cos x - \cos^2 x = \sin x + \sin^2 x \Rightarrow 2\sin x \cdot \cos x + 2\cos x = \sin x + \sin^2 x + \cos^2 x \Rightarrow$ $2\sin x \cdot \cos x + 2\cos x = \sin x + 1 \Rightarrow 2\cos x(\sin x + 1) = 1(\sin x + 1) \Rightarrow$
$(2\cos x - 1)(\sin x + 1) = 0 \Rightarrow \cos x = \frac{1}{2}$ or $\sin x = -1 \Rightarrow x = 60°, 270°, 300°$.

3. $\sin x = \frac{1}{2}\sqrt{8\cos x + 7} \Rightarrow \sin^2 x = \frac{1}{4}(8\cos x + 7) \Rightarrow 4(1 - \cos^2 x) = 8\cos x + 7 \Rightarrow$
$4\cos^2 x + 8\cos x + 3 = 0 \Rightarrow (2\cos x + 1)(2\cos x + 3) = 0 \Rightarrow \cos x = -\frac{1}{2}, \cancel{-\frac{3}{2}} \Rightarrow$
x = 120°, 240°, but $\sin 240° < 0 \Rightarrow$ only solution is 120°.

TEAM ROUND

1. $72 = 2^3 \cdot 3^2$; if n was prime $\Rightarrow$ $72n$ would have $4\times3\times2 = 24$ factors; since n is composite $\Rightarrow$ n must just consist of factors of 2 and/or 3. $2^5 \cdot 3^3$ has 6×4 factors $\Rightarrow n = 2^2 \cdot 3 = 12$; $2^3 \cdot 3^5$ has 4×6 factors $\Rightarrow n = 3^3 = 27$; $2^7 \cdot 3^2$ has 8×3 factors $\Rightarrow n = 2^4 = 16$; therefore, n = 12, 16, or 27.

2. Let $m\angle QPT = m\angle TPS = m\angle RPS = x$ and
let $m\angle R = y \Rightarrow m\angle PQU = \frac{1}{2}y$;
$3x + 2y = 180$ and $2x + \frac{1}{2}y = 105 \Rightarrow$
$8x + 2y = 420 \Rightarrow 5x = 240 \Rightarrow x = 48 \Rightarrow$
$y = 18 \Rightarrow m\angle UVT = 180 - x - \frac{1}{2}y = 123$.



3. Let n = number of nickels, d = number of dimes, q = number of quarters $\Rightarrow$
$5n + 10d + 25q = 785$ and $q = n + 9 \Rightarrow n + 2d + 5q = 157$ and $q = n + 9 \Rightarrow$
$n + 2d + 5(n + 9) = 157 \Rightarrow 6n + 2d = 112 \Rightarrow 3n + d = 56 \Rightarrow$ If d = 2, n = 18, and q = 27 for a total of 47 coins; If d = 53, n = 1, and q = 10 for a total of 64 coins; $64 - 47 = 17$.

Note: As the number of nickels go up by 1, the number of dimes go down by 3, and the number of quarters go up by one, giving you one less coin than before. So the extreme values for the number of nickels (1 and 18) give the largest and smallest number of coins.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ANSWER SHEET:

ROUND 1

1. 102
2. 0, 6
3. 9

ROUND 2

1. $(5,1)$
2. 1100 ($1100)
3. 8

ROUND 3

1. 22.5 or equivalent (22.5°)
2. 44 (44°)
3. 117 (117°)

ROUND 4

1. 1440
2. –12, 6
3. 16 (16 miles per hour)

ROUND 5

1. 30°, 330°
2. 60°, 270°, 300°
3. 120°

TEAM ROUND

3 pts. 1. 12, 16, 27

3 pts. 2. 123 (123°)

4 pts. 3. 17

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2003

ROUND 1 – Arithmetic-Open

1. ___________________

2. ___________________

3. ___________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the smallest whole number with exactly 8 whole number factors.

2. Find the rational number $\frac{a}{b}$ in base 10 which equals $0.66\overline{6}_{(9)} + 0.2_{(3)} + 0.12_{(6)}$.

3. How many 3-digit whole numbers are such that each of the numbers and 21 are relatively prime?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2003

ROUND 2 – Simultaneous Linear Equations, Word Problems, Matrices

1. ________________

2. (____,____)

3. ________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The length of one side of a square is 2 cm less than 6 times the length of one side of an equilateral triangle. The sum of their perimeters is 37cm. Find the number of centimeters in the length of one side of the equilateral triangle.

2. If $\begin{pmatrix} 2x & 3y \\ -x & 4y \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}=\begin{pmatrix} 10-6x \\ 9y+3 \end{pmatrix}$, find the ordered pair (x,y)

3. On Monday, Jose walked 5 miles and ran 3 miles in a total time of 2 hours and 30 minutes. On Tuesday, Jose walked 4 miles and ran 4 miles in a total time of 2 hours and 16 minutes. Find the number of miles per hour in Jose's walking rate.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2003

ROUND 3 – Geometry: Angles and Triangles

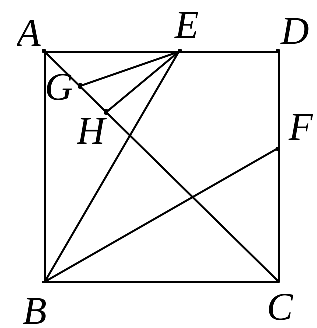
1. ____________________

2. ____________________

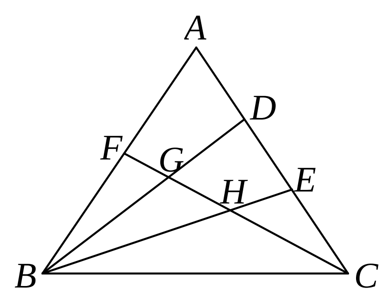
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given square $ABCD$, $\overline{BE}$ and $\overline{BF}$ trisect $\angle ABC$, and $\overline{EG}$ and $\overline{EH}$ trisect $\angle AEB$, find the ratio of $m\angle EGH$ to $m\angle EHG$.



2. In ΔABC, $AB = AC$, $m\angle A : m\angle ABC = 4:3$, $\overline{BD}$ and $\overline{BE}$ trisect $\angle ABC$ and $\overline{CF}$ bisects $\angle ACB$. Find the ratio of $m\angle BGC$ to $m\angle FHE$.



3. Given one regular polygon has 6 more sides than another and the measures of one interior angle of each polygon differ by 10°, find the number of sides of both regular polygons.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2003

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Solve the following equation over real numbers: $\left(x^2+2x\right)^2=18\left(x^2+2x\right)-45$.

2. A right triangle has one leg 5 inches longer than the other. A square is constructed having each side equal to the length of the hypotenuse of the right triangle. The area of the square is 17 square inches less than 5 times the area of the triangle. Find the number of inches in the sum of the two legs of the right triangle.

3. The quadratic equation in *x*, $x^2+bx+c=0$, has roots *r* and *s*. The quadratic equation in *x*, $x^2+cx+b=0$, has roots $\frac{2}{r}$ and $\frac{2}{s}$. Solve for *c* over real numbers.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2003

ROUND 5 – Trig. Equations

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given $0° \le x < 360°$, and $\tan x = 2\sin x$, find all solutions for *x*.

2. Given $0° \le x < 360°$ and $\sin 3x \cos 2x = 1 - \cos 3x \sin 2x$, find the <u>sum</u> of all solutions for *x*.

3. Given $\sin(90° + x)\cos(180° + x) + \sec 300° \cos(270° + x) = \csc 210° \csc(x - 180°)$ and $0° \le x < 360°$, find all solutions for *x*.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2003

TEAM ROUND Time limit: 12 minutes

3 pts. 1. ________________

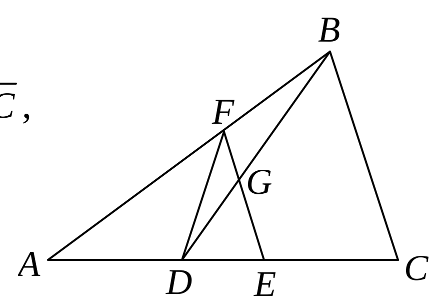
3 pts. 2. ________________

4 pts. 3. (____ , ____)

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Al picked an odd natural number greater than 1, squared it, subtracted 1 from the result, and then divided this second result by 4. He noted this final result was divisible by 77. What is the smallest possible odd natural number Al could have originally picked?

2. Given $AB = AC$, $AD = DF = EF = CE$, and $\overline{EF} \parallel \overline{BC}$, find the number of degrees in $m\angle BGE$.



3. Given parameters a and b and the following system of linear equations with more than one solution, find the ordered pair (a,b).

$$\begin{cases} 4x + 3y + z = -14 \\ 3x + ay = -11 \\ 5x + 4y + 2z = b \end{cases}$$

Detailed Solutions of GBML MEET 2 – NOVEMBER 2003

ROUND 1

1. A whole number with exactly 8 factors is of the form p^7, p^3q or pqr where p, q, and r are distinct primes. Since $2^3 \times 3 = 24 < 2 \times 3 \times 5 = 30$, the answer is 24.

2. $0.66\overline{6}_{(9)} + 0.2_{(3)} + 0.12_{(6)} = \dfrac{\frac{6}{9}}{1-\frac{1}{9}} + \dfrac{2}{3} + \dfrac{1}{6} + \dfrac{2}{36} = \dfrac{3}{4} + \dfrac{8}{9} = \dfrac{59}{36}$.

3. There are 900 3-digit whole numbers; of these 3×34 to 3×333, which are 300 numbers divisible by 3, 7×15 to 7×142, which are 128 numbers divisible by 7, and 21×5 to 21×47, which are 43 numbers divisible by 21; there are 300 + 128 – 43 = 385 numbers divisible by 3 or 7 ⇒ there are 900 – 385 = 515 numbers relatively prime with 21.

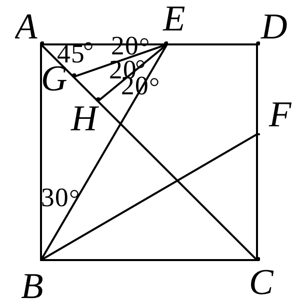
ROUND 2

1. Let x = length of one side of the triangle and y = length of one side of the square:
$y = 6x - 2$ and $4y + 3x = 37 \Rightarrow 24x - 8 + 3x = 37 \Rightarrow 27x = 45 \Rightarrow x = \dfrac{5}{3}$.

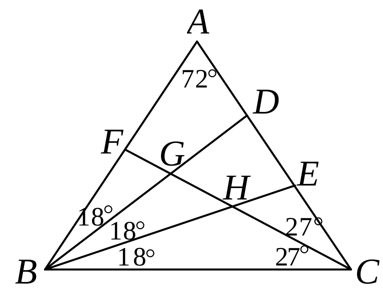
2. $\begin{pmatrix} 2x & 3y \\ -x & 4y \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 10-6x \\ 9y+3 \end{pmatrix} \Rightarrow \begin{cases} 10x + 6y = 10 - 6x \\ -5x + 8y = 9y + 3 \end{cases} \Rightarrow \begin{cases} 16x + 6y = 10 \\ -5x - y = 3 \end{cases} \Rightarrow$
$\begin{cases} 16x + 6y = 10 \\ -30x - 6y = 18 \end{cases} \Rightarrow -14x = 28 \Rightarrow x = -2 \Rightarrow 10 - y = 3 \Rightarrow y = 7 \Rightarrow (-2,7)$ is the answer.

3. Let x = Jose's walking rate and let y = Jose's running rate:
$\begin{cases} \dfrac{5}{x} + \dfrac{3}{y} = \dfrac{5}{2} \\ \dfrac{4}{x} + \dfrac{4}{y} = \dfrac{34}{15} \end{cases} \Rightarrow \begin{cases} \dfrac{20}{x} + \dfrac{12}{y} = 10 \\ \dfrac{-12}{x} + \dfrac{-12}{y} = \dfrac{-34}{5} \end{cases} \Rightarrow \dfrac{8}{x} = \dfrac{16}{5} \Rightarrow x = 2.5 \text{ mph}$

ROUND 3



1. Since right $\angle ABC$ is trisected $\Rightarrow m\angle ABE = 30° \Rightarrow m\angle AEB = 60° \Rightarrow m\angle AEG = m\angle GEH = m\angle HEB = 20° \Rightarrow m\angle EGH = 45° + 20° = 65°$ and $m\angle EHG = 180° - 45° - 40° = 95°$; $\dfrac{m\angle EGH}{m\angle EHG} = \dfrac{65}{95} = \dfrac{13}{19}$.



2. Let $m\angle A = 4x \Rightarrow m\angle ABC = m\angle ACB = 3x \Rightarrow 10x = 180 \Rightarrow x = 18 \Rightarrow m\angle A = 72$, $m\angle ADB = m\angle DBE = m\angle EBC = 18°$ and $m\angle ACF = m\angle BCF = 27°$; $m\angle BGC = 180° - 36° - 27° = 117°$; $m\angle FHE = m\angle BHC = 180° - 18° - 27° = 135°$; $\dfrac{117}{135} = \dfrac{13}{15}$.

3. Let x and $x+6$ be the number of sides of the regular polygons:
$\left(180 - \dfrac{360}{x+6}\right) - \left(180 - \dfrac{360}{x}\right) = 10 \Rightarrow \dfrac{360}{x} - \dfrac{360}{x+6} = 10 \Rightarrow \dfrac{36}{x} - \dfrac{36}{x+6} = 1 \Rightarrow$
$36(x+6) - 36x = x(x+6) \Rightarrow x^2 + 6x - 36 \cdot 6 = 0 \Rightarrow (x-12)(x+18) = 0 \Rightarrow$
$x = 12 \Rightarrow$ the two regular polygons have 12 and 18 sides.

ROUND 4

1. $(x^2+2x)^2 = 18(x^2+2x) - 45 \Rightarrow (x^2+2x)^2 - 18(x^2+2x) + 45 = 0 \Rightarrow$
$(x^2+2x-3)(x^2+2x-15) = 0 \Rightarrow (x+3)(x-1)(x+5)(x-3) = 0 \Rightarrow x = -5, -3, 1, 3$

2. Let the legs of the right triangle = x and $x+5 \Rightarrow$ hypotenuse = $\sqrt{(x+5)^2 + x^2} =$
$\sqrt{2x^2+10x+25} \Rightarrow \left(\sqrt{2x^2+10x+25}\right)^2 = 5 \cdot \dfrac{1}{2}x(x+5) - 17 \Rightarrow$
$2(2x^2+10x+25) = 5x(x+5) - 34 \Rightarrow 4x^2+20x+50 = 5x^2+25x-34 \Rightarrow$
$x^2+5x-84 = 0 \Rightarrow (x+14)(x-7) = 0 \Rightarrow x = 7 \Rightarrow$ sum of legs = 7 + 12 = 19.

3. From the 1st equation, $rs = c$ and $r+s = -b$; from the 2nd equation, $\dfrac{4}{rs} = b$ and
$\dfrac{2}{r} + \dfrac{2}{s} = -c \Rightarrow \dfrac{2(r+s)}{rs} = -c \Rightarrow \dfrac{2(-b)}{c} = -c \Rightarrow c^2 = 2b$ and $b = \dfrac{4}{c} \Rightarrow$
$c^2 = \dfrac{8}{c} \Rightarrow c^3 = 8 \Rightarrow c = 2$.

ROUND 5

1. $\tan x = 2\sin x \Rightarrow \dfrac{\sin x}{\cos x} = 2\sin x \Rightarrow 2\sin x\cos x - \sin x = 0$ and $\cos x \neq 0 \Rightarrow$

$\sin x(2\cos x - 1) = 0 \Rightarrow \sin x = 0$ or $\cos x = \dfrac{1}{2} \Rightarrow x = 0°,\ 60°, 180°,\ 300°$.

2. $\sin 3x \cos 2x = 1 - \cos 3x \sin 2x \Rightarrow \sin 3x \cos 2x + \cos 3x \sin 2x = 1 \Rightarrow$
$\sin(3x + 2x) = 1 \Rightarrow \sin 5x = 1 \Rightarrow 5x = 90°, 450°, 810°, 1170°, 1530° \Rightarrow$ sum of the solutions for $x = \dfrac{90° + 450° + 810° + 1170° + 1530°}{5} = 810°$.

3. $\sin(90° + x)\cos(180° + x) + \sec 300° \cos(270° + x) = \csc 210° \csc(x° - 180°) \Rightarrow$ by reduction formulas, $\cos x(-\cos x) + 2\sin x = -2(-\csc x) \Rightarrow -\cos^2 x + 2\sin x = \dfrac{2}{\sin x} \Rightarrow$

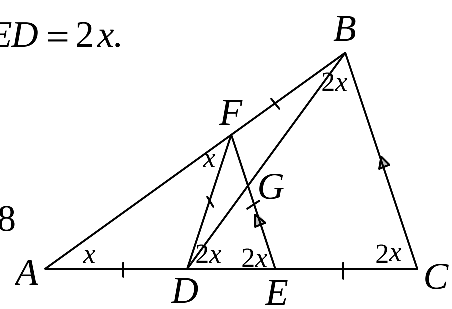
$\sin x \neq 0$ and $-\cos^2 x \sin x + 2\sin^2 x = 2 \Rightarrow \cos^2 x \sin x + 2(1 - \sin^2 x) = 0 \Rightarrow$

$\Rightarrow \cos^2 x \sin x + 2\cos^2 x = 0 \Rightarrow \cos^2 x(\sin x + 2) = 0 \Rightarrow \cos x = 0 \Rightarrow x = 90°,\ 270°$.

TEAM ROUND

1. $\dfrac{(2x+1)^2 - 1}{4} = x^2 + x = x(x+1)$; the smallest value for x which makes this result divisible by 77 is $x = 21$. (21×22 is divisible by 77.) Therefore the odd number picked was $2 \times 21 + 1 = 43$.

2. Let $m\angle A = x \Rightarrow m\angle AFD = x \Rightarrow m\angle FDE = 2x \Rightarrow m\angle FED = 2x$.
By corresponding angles $m\angle C = 2x \Rightarrow m\angle ABC = 2x \Rightarrow$
$BCEF$ is an isosceles trapezoid $\Rightarrow CE = BF \Rightarrow BF = DF$.
$5x = 180 \Rightarrow x = 36$; $m\angle BFE = 180 - 2x = 108$.
$m\angle DFB = 180 - x = 144 \Rightarrow m\angle FBD = (180 - 144) \div 2 = 18$
$\Rightarrow m\angle BGE = m\angle BFE + m\angle FBD = 108 + 18 = 126$.



3. The system has infinite solutions $\Rightarrow$ some linear combination of two of the equations will produce an equation equivalent to the 3rd. Multiply the first equation by 2 and subtract the third equation: $3x + 2y = -28 - b \Rightarrow a = 2$ and $-28 - b = -11 \Rightarrow b = -17 \Rightarrow$ $(a, b) = (2, -17)$. An alternative, but longer solution would be using the result that the 3 by 3 determinant of the system must equal 0 (and now solve for a) and then when you replace any column with the constant terms, that 3 by 3 determinant must also = 0 (and now solve for b).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2003

ANSWER KEY:

ROUND 1

1. 24
2. $\frac{59}{36}$
3. 515

ROUND 2

1. $\frac{5}{3}$ $\left(\frac{5}{3}\text{cm}\right)$
2. $(-2,7)$
3. 2.5 or equivalent (2.5 mph)

ROUND 3

1. 13:19 $\left(\frac{13}{19}\right)$
2. 13:15 $\left(\frac{13}{15}\right)$
3. 12, 18

ROUND 4

1. –5, –3, 1, 3 (in any order)
2. 19 (19 inches)
3. 2

ROUND 5

1. 0°, 60°, 180°, 300° (in any order)
2. 810°
3. 90°, 270°

TEAM ROUND

3 pts. 1. 43

3 pts. 2. 126 (126°)

4 pts. 3. $(2,-17)$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

ROUND 1 – Algebra 1: Fractions and Word Problems

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Simplify $\dfrac{2x + 7}{x^2 + 2x + 4} - \dfrac{x^2 + 3x + 26}{8 - x^3}$

2. The rate of flow of water into a pool via an inlet pipe is twice the rate out via a drain. When the pool is one quarter full, the drain is opened for eight minutes and then closed. The inlet pipe is now opened and fills the pool in forty minutes. How many minutes would it take the inlet pipe to fill the pool if it was empty to begin with and the drain is kept closed?

3. Arthur and Betsy are painting a house. One day Arthur painted for six hours, and Betsy painted for two hours and one-third of the house was painted. The next day Arthur painted for three hours and Betsy painted for six hours, and the house was completely painted. Assuming each one works at his or her own constant rate throughout, how long in hours would it take Arthur by himself to paint the entire house?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

ROUND 2 – Coordinate Geometry of the Straight Line

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given line $l: 3y - 4mx = 6m$ with the sum of its x intercept and y intercept equaling – 6, find all values of m which satisfy these conditions.

2. Given the three lines $l_1 : y = (3k + 2)x + 1$, $l_2 : y = (6k + 1)x + 2$, and $l_3 : y = mx + 3$, where l_2 and l_3 are parallel and l_1 and l_2 are perpendicular, find all possible values for m.

3. Find the area bounded by the y axis and the lines $3x - 2y = 12$ and $3x + 4y = 48$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

ROUND 3 – Geometry: Polygons: Area and Perimeter

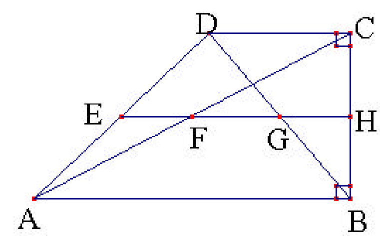
1. ____________________

2. ____________________

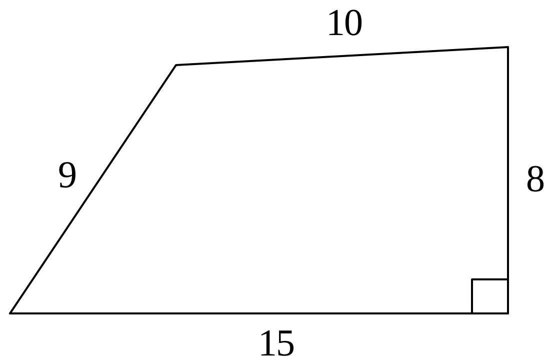
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

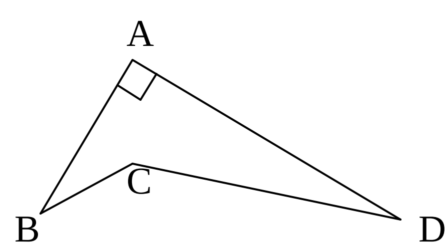
1. Given quadrilateral ABCD with right angles at vertices B and C as indicated on the figure, BC = 8, E and H midpoints of $\overline{AD}$ and $\overline{BC}$ respectively, EF = 2.5, and FG = 3, find the area of quadrilateral ABCD.



2. Find the area of this quadrilateral which has only 1 right angle as indicated on the diagram.



3. Find the area of this quadrilateral ABCD such that AB = 6, AD = $6\sqrt{3}$, BC = $2\sqrt{3}$, $\angle$ A is right, and m $\angle$ B = 30°.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

ROUND 4 – Algebra 2– Logs, Exponents, Radicals and equations involving them

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Solve the following equation for x: $\log_6(2x-1)+\log_6(3x-11)=2$

2. If $\log_b 24 = x$ and $\log_b \sqrt[3]{\frac{2}{9}} = y$, find $\log_b 2$ in terms of x and y.

3. Any solution to the equation, $\frac{\log_3 x}{\log_x 3} = \log_9 \sqrt[3]{x^4} + \log_8 16^2$, can be put in the form 3^a. Find all possible values for a.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

ROUND 5 – Trig. analysis and Complex Numbers, Trig Form

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Put $\dfrac{\left(-2+2i\sqrt{3}\right)^5}{\left(\sqrt{2}+i\sqrt{2}\right)^7}$ in the form $r\ cis\ \theta$ where $r > 0$ and $0^\circ \le \theta < 360^\circ$.

2. Given $\cos A = \dfrac{\sqrt{2}}{3}$ and $\tan A < 0$, find the value for $\csc\left(180^\circ - A\right)\tan\left(90^\circ + A\right)$.

3. Solve the following equation over the complex numbers: $z^3 i\sqrt{2} = 125 - 125i$ and put all values for z in the form $r\ cis\ \theta$ where $r > 0$ and $0^\circ \le \theta < 360^\circ$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

TEAM ROUND

3 pts. 1. ___________________

3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. The reciprocal of one more than the reciprocal of a number is k more than twice the number. If this number is **unique**, then the values for k can be put in the form $a \pm \sqrt{b}$. Find the sum $a + b$.

2. Given points O, the origin, A(0, 4), B(0, 6), and C(8, 0), the bisector of $\angle$ OBC and $\overline{AC}$ intersect at point D. Find the area of Δ BCD.

3. An isosceles triangle has two of its vertices on the positive x axis and its third vertex at (6, 4). If the slope of one of its legs is $\frac{4}{3}$, find all possible lines in the form Ax + By = C, where A, B, and C are relatively prime integers and **A > 0**, that contain (6, 4), do **not** have a slope of $\frac{4}{3}$, and contain a side of the isosceles triangle.

Detailed Solutions of GBML for MEET 3 – DECEMBER 1998

ROUND 1

1. $$\frac{2x+7}{x^2+2x+4} - \frac{x^2+3x+26}{8-x^3} = \frac{2x+7}{x^2+2x+4} + \frac{x^2+3x+26}{x^3-8} =$$

$$\frac{(x-2)(2x+7)}{(x-2)(x^2+2x+4)} + \frac{x^2+3x+26}{(x-2)(x^2+2x+4)} = \frac{2x^2+3x-14+x^2+3x+26}{(x-2)(x^2+2x+4)} =$$

$$\frac{3x^2+6x+12}{(x-2)(x^2+2x+4)} = \frac{3(x^2+2x+4)}{(x-2)(x^2+2x+4)} = \mathbf{\frac{3}{x-2}}$$

2. The time to fill or empty the tank is inversely proportional to the rate of flow $\Rightarrow$
If x = time to fill the pool, then $2x$ = time to empty the pool $\Rightarrow$
Equation: $\frac{40}{x} - \frac{8}{2x} = \frac{3}{4} \Rightarrow \frac{40}{x} - \frac{4}{x} = \frac{3}{4} \Rightarrow \frac{36}{x} = \frac{3}{4} \Rightarrow x =$ **48 min.**

3. Let a = Arthur's rate, b = Betsy's rate;
$\frac{6}{a} + \frac{2}{b} = \frac{1}{3}; \frac{3}{a} + \frac{6}{b} = \frac{2}{3} \Rightarrow \frac{1}{a} + \frac{2}{b} = \frac{2}{9} \Rightarrow$
$\frac{5}{a} = \frac{1}{9} \Rightarrow \Rightarrow$ ***a*** = **45** hours.

ROUND 2

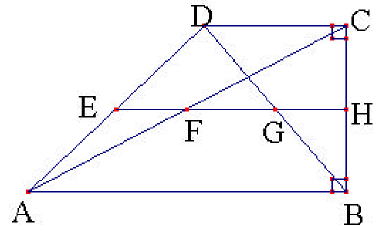
1. $l: 3y - 4mx = 6m$: If $x = 0$, then $y = 2m$; if $y = 0$, then $x = -\frac{3}{2} \Rightarrow 2m - \frac{3}{2} = -6 \Rightarrow m = \mathbf{-\frac{9}{4}}$

2. $(3k+2)(6k+1) = -1 \Rightarrow 18k^2 + 15k + 3 = 0 \Rightarrow 6k^2 + 5k + 1 = 0 \Rightarrow$
$(3k+1)(2k+1) = 0 \Rightarrow k = -\frac{1}{2}$ or $-\frac{1}{3} \Rightarrow m = 6\left(-\frac{1}{3}\right) + 1$ or $6\left(-\frac{1}{2}\right) + 1 =$ **-1 or –2**

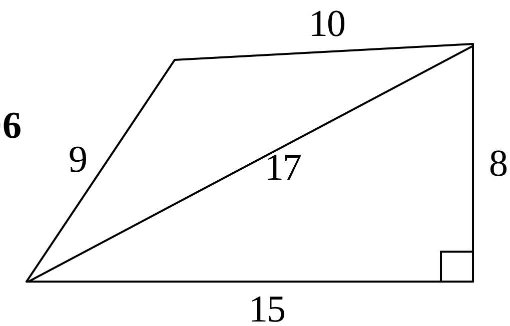
3. The y intercepts of the two lines are – 6 and 12. To find the x coordinate of the point of intersection: $6x - 4y = 24$ and $3x + 4y = 48 \Rightarrow 9x = 72 \Rightarrow x = 8$. The area of the triangle $= \frac{1}{2}(12 - (-6))8 =$ **72**

ROUND 3

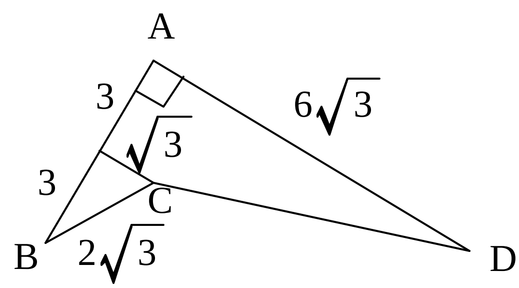
1. ABCD is a trapezoid $\Rightarrow$ Area = $m\,h$, where m is the median and h is its height, EF = $\frac{1}{2}$DC = GH $\Rightarrow$ EH = 2.5 + 3 + 2.5 = 8 = m $\Rightarrow$ Area = **64**



2. The area of the right triangle = $\left(\frac{1}{2}\right)(15)(8) = 60$
Use Hero's formula to find the other triangle's area:
$s = 18: \sqrt{18 \cdot 1 \cdot 8 \cdot 9} = 36 \Rightarrow$ Area = 60 + 36 = **96**



3. Draw a perpendicular from C to $\overline{AB}$. The lengths of the sides of the 30-60-90° Δ are indicated on the diagram. The quadrilateral is now divided into a trapezoid and a triangle. Area of triangle = $\frac{3\sqrt{3}}{2}$; Area of trapezoid = $\frac{21\sqrt{3}}{2} \Rightarrow$ Total area = $\mathbf{12\sqrt{3}}$



ROUND 4

1. $\log_6(2x-1) + \log_6(3x-11) = 2 \Rightarrow (2x-1)(3x-11) = 36 \Rightarrow$
$6x^2 - 25x - 25 = 0 \Rightarrow (6x+5)(x-5) = 0 \Rightarrow x = 5$ only since $x = -\frac{5}{6}$ does not check $\Rightarrow$
Solution = **5**

2. $\log_b 24 = x$ and $\log_b \sqrt[3]{\frac{2}{9}} = y \Rightarrow 3\log_b 2 + \log_b 3 = x$ and $\log_b 2 - 2\log_b 3 = 3y \Rightarrow$
$6\log_b 2 + 2\log_b 3 = 2x$ and $\log_b 2 - 2\log_b 3 = 3y \Rightarrow 7\log_b 2 = 2x + 3y \Rightarrow \log_b 2 = \mathbf{\frac{2x+3y}{7}}$

3. $\dfrac{\log_3 x}{\log_x 3} = \log_9 \sqrt[3]{x^4} + \log_8 16^2 \Rightarrow \dfrac{\frac{\log x}{\log 3}}{\frac{\log 3}{\log x}} = \dfrac{\frac{4}{3}\log x}{2\log 3} + \dfrac{\log\left(2^4\right)^2}{3\log 2} \Rightarrow \left(\log_3 x\right)^2 - \frac{2}{3}\log_3 x - \frac{8}{3} = 0$
Let $a = \log_3 x \Rightarrow 3a^2 - 2a - 8 = 0 \Rightarrow (3a+4)(a-2) = 0 \Rightarrow a = \mathbf{-\frac{4}{3}}$ **or 2**

ROUND 5

1. $-2 + 2i\sqrt{3} = 4\ cis\ 120°$ and $\sqrt{2} + i\sqrt{2} = 2\ cis\ 45° \Rightarrow$
$$\frac{\left(-2+2i\sqrt{3}\right)^5}{\left(\sqrt{2}+i\sqrt{2}\right)^7} = \frac{\left(4\ cis\ 120°\right)^5}{\left(2\ cis\ 45°\right)^7} = \frac{2^{10}\ cis\ 600°}{2^7\ cis\ 315°} = \mathbf{8\ cis\ 285°}$$

2. A is a rotation into quadrant IV and $\cos A = \frac{\sqrt{2}}{3} \Rightarrow x = \sqrt{2};\ y = -\sqrt{7}\ ;\ r = 3$

By the reduction formulas, $\csc\left(180° - A\right) = \csc A = -\frac{3}{\sqrt{7}}$ and

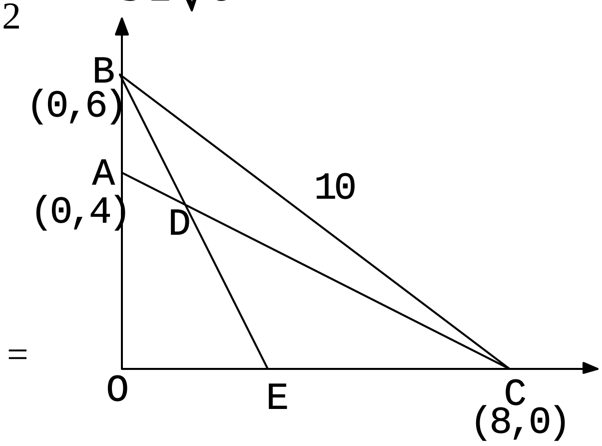
$\tan\left(90° + A\right) = -\cot A = \frac{\sqrt{2}}{\sqrt{7}} \Rightarrow \csc\left(180° - A\right)\tan\left(90° + A\right) = \mathbf{-\frac{3\sqrt{2}}{7}}$

3. $z^3 i\sqrt{2} = 125 - 125i \Rightarrow z^3\left(\sqrt{2}\,cis\ 90°\right) = 125\sqrt{2}\ cis\ 315° \Rightarrow z^3 = 125\ cis\ 225° \Rightarrow$
$n = 0,1,2: z = 5\ cis\left(\frac{225°}{3} + 120°n\right) \Rightarrow z = \mathbf{5\ cis\ 75°,\ 5\ cis\ 195°,\ 5\ cis\ 315°}$

TEAM ROUND

1. $\frac{1}{\frac{1}{x}+1} = 2x + k \Rightarrow \frac{x}{1+x} = 2x + k \Rightarrow 2x^2 + \left(k+1\right)x + k = 0$. For there to be one solution for $x \Rightarrow \left(k+1\right)^2 - 8k = 0 \Rightarrow k^2 - 6k + 1 = 0 \Rightarrow k = \frac{6 \pm \sqrt{32}}{2} = 3 \pm \sqrt{8} \Rightarrow a + b = \mathbf{11}$

2. By the angle bisector theorem,
$OE:EC = 6:10 = 3:5 \Rightarrow E = (3, 0) \Rightarrow$
line $\overline{BE}$ is $y = -2x + 6$ and line $\overline{AC}$ is $y = -\frac{1}{2}x + 4$
Finding coordinates of D:
$-2x + 6 = -\frac{1}{2}x + 4 \Rightarrow x = \frac{4}{3}$ and $y = \frac{10}{3}$
Area of Δ BCD = area of Δ BCE – area of Δ DCE =
$\left(\frac{1}{2}\right)\left(5\right)\left(6\right) - \left(\frac{1}{2}\right)\left(5\right)\left(\frac{10}{3}\right) = \mathbf{\frac{20}{3}}$



3. If the base of the isosceles triangles is along the x axis then the line would have slope $-\frac{4}{3}$
$\Rightarrow y - 4 = -\frac{4}{3}\left(x - 6\right) \Rightarrow 4x + 3y = 36$. To find another possibility $y - 4 = \frac{4}{3}\left(x - 6\right) \Rightarrow$ when $y = 0$, $x = 3$. The distance from (3, 0) to (6, 4) = 5 $\Rightarrow$ the line would intersect the x axis at (8, 0) $\Rightarrow$ the line through (8, 0) and (6,4) is $2x + y = 16 \Rightarrow$ Lines are
$\mathbf{4x + 3y = 36}$ or $\mathbf{2x + y = 16}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

ANSWER SHEET:

ROUND 1

1. $\frac{3}{x-2}$ $\left(\frac{-3}{2-x}\right)$
2. 48
3. 45

ROUND 2

1. $-\frac{9}{4}$
2. -1, -2
3. 72

ROUND 3

1. 64
2. 96
3. $12\sqrt{3}$

ROUND 4

1. 5
2. $\frac{2x+3y}{7}$
3. $-\frac{4}{3}$, 2

ROUND 5

1. 8 *cis* 285°
2. $-\frac{3\sqrt{2}}{7}$
3. 5 *cis* 75°, 5 *cis* 195°, 5 *cis* 315°

TEAM ROUND

3 pts. 1. 11

3 pts. 2. $\frac{20}{3}$

4 pts. 3. $4x + 3y = 36$, $2x + y = 16$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

ROUND 1 – Algebra 1: Fractions and Word Problems

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Solve the following equation for *x*: $\dfrac{18}{x^2-9}+\dfrac{2}{3-x}=\dfrac{x}{x+3}$

2. Bob can type a long dictated manuscript in 18 hours while Alice can type it in 15 hours. Alice and Bob work for 6 hours typing different parts of the manuscript and then stop. Charles types the rest of the manuscript in 8 hours. How many hours would it have taken Charles and Bob, working together, to type the entire manuscript?

3. The rational expression, $\dfrac{x^2-2x-15}{3x^2-5x+k}$, can be reduced for what value(s) of k.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

ROUND 2 – Coordinate Geometry of the Straight Line

1. (,)

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given line L: $\{(x, y): 2x - 3y - 24 = 0\}$, find the coordinates of the midpoint of the line segment cut off line L by the coordinate axes.

2. If line λ is the reflection of the line $3x - 2y + 8 = 0$ about the line $x = 4$, find the y-intercept of line λ.

3. If lines L_1: $ax + 3y = 31$, L_2: $5x - 2y = 26$, and L_3: $3x - 4y = 24$ are concurrent, find the area of the triangle formed by lines L_1, L_2 and the x axis.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

ROUND 3 – Geometry: Polygons: Area and Perimeter

1. ____________________

2. ____________________

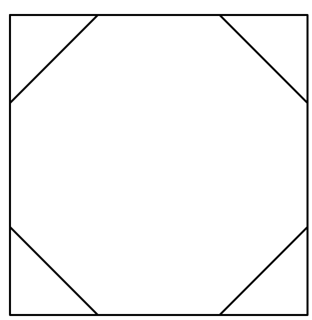
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A regular hexagon has a perimeter of $12\sqrt{3}$ inches. An equilateral triangle is constructed that has a side equal in length to one of the longest diagonals of the hexagon. Find the number of square inches in the area of this equilateral triangle.

2. A triangle has sides of length 8, 15, and 17 centimeters. From a point in the interior of the triangle perpendiculars are drawn to all three sides. If the perpendicular drawn to the longest side is 2 cm., and the perpendicular drawn to the shortest side is 4 cm., find the length in centimeters of the perpendicular drawn to the remaining side.

3. Four congruent right isosceles triangles are sliced off each corner of a square leaving a regular octagon. If the area of the octagon is 4 square units, find the area of the original square.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

ROUND 4 – Algebra 2– Logs, Exponents, Radicals and equations involving them

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. If $\left(3+\sqrt{2}\right)^2 - \sqrt[4]{4} + \dfrac{2}{\sqrt{2}-1} = x + y\sqrt{2}$, where x and y are rational, find the sum, $x + y$.

2. Solve for x: $\log_7\left(x^3 - 27\right) = -\log_7\left(x-3\right) = 2$

3. If $\log_{a^2} b + \log_{a^3} 2b = \log_a \sqrt[6]{x}$, solve for x in terms of b.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

ROUND 5 – Trig. analysis and Complex Numbers, Trig Form

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. If $\sin q = \frac{3}{4}$ and $\cos q < 0$, compute $\tan\left(90^\circ + q\right)$.

2. Find the **<u>smallest two positive degree measures</u>** for q satisfying the equation:
$\cos q = 2\sin 19.5^\circ \cos 19.5^\circ$

3. Given z is a second quadrant point in the complex plane, z is a solution to the equation, $z^3 = 13.5 - 13.5i\sqrt{3}$, and $zw = -6\sqrt{2} - 6i\sqrt{2}$, solve for w in the polar form $r\,\text{cis}\,q$, where $r > 0$ and $0^\circ \le q < 360^\circ$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

TEAM ROUND

3 pts. 1. ____________________

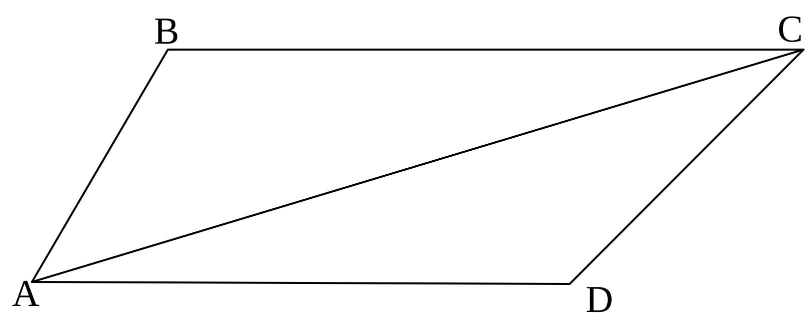
3 pts. 2. ____________________

4 pts. 3. ____________________

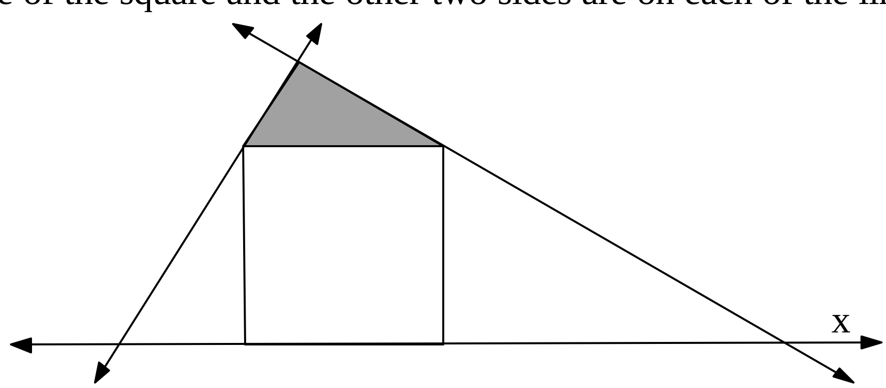
SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. If $\log_3 12 - \log_2 18 = a$ and $\log_2 3 = b$, find a in terms of b.

2. Given trapezoid ABCD, m $\angle$ BAD = 60°, m $\angle$ D = 135°, AD = 30, and CD = $8\sqrt{6}$, find the exact area of Δ ABC.



3. Given the lines $2x - y + 9 = 0$ and $x + 3y - 6 = 0$, a square is constructed with one side along the x axis and the other sides as shown. Find the area of the shaded triangle one of whose sides is a side of the square and the other two sides are on each of the lines.



Detailed Solutions of GBML for MEET 3 – DECEMBER 1999

ROUND 1

1. $\frac{18}{x^2-9}+\frac{2}{3-x}=\frac{x}{x+3} \Rightarrow \frac{18}{x^2-9}-\frac{2}{x-3}=\frac{x}{x+3} \Rightarrow 18-2(x+3)=x(x-3) \Rightarrow$ $18-2x-6=x^2-3x \Rightarrow x^2-x-12=0 \Rightarrow (x-4)(x+3)=0 \Rightarrow x=4$ since –3 is an extraneous solution to the original equation.

2. $\frac{6}{15}+\frac{6}{18}=\frac{11}{15} \Rightarrow$ Charles has $\frac{4}{15}$ of the manuscript left to type. $\Rightarrow \frac{4}{15}=\frac{8}{x} \Rightarrow x=30$ hrs $\frac{x}{18}+\frac{x}{30}=1 \Rightarrow x=\frac{45}{4}$ *or* 11.25 *hrs.*

3. $\frac{x^2-2x-15}{3x^2-5x+k} = \frac{(x-5)(x+3)}{3x^2-5x+k}$; this is reducible only if either 5 or –3 are zeros of the bottom polynomial $\Rightarrow 3(5)^2-5(5)+k=0$ *or* $3(-3)^2-5(-3)+k=0 \Rightarrow k=-42$ *or* -50

ROUND 2

1. $x=0: -3y-24=0 \Rightarrow y=-8;\ y=0: 2x-24=0 \Rightarrow x=12;$ The midpoint of the segment whose endpoints are (0, -8) and (12, 0) is (6, -4).

2. $3x-2y+8=0$ when $x=4 \Rightarrow y=10$; the slope of the line is $\frac{3}{2} \Rightarrow$ the slope of the reflection of this line = $-\frac{3}{2} \Rightarrow$ the equation of the line is: $y-10=-\frac{3}{2}(x-4)$; the y-intercept of this line is 16.

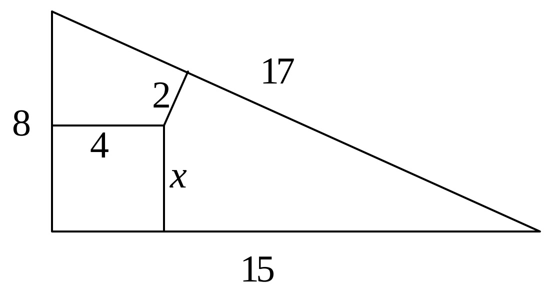
3. Since $ax+3y=31$, L_2: $5x-2y=26$, and L_3: $3x-4y=24$ are concurrent, find the intersection of L_2 and L_3; $-10x+4y=-52 \Rightarrow -7x=-28 \Rightarrow x=4$ *and* $y=-3 \Rightarrow$ $4a-9=31 \Rightarrow a=10$; the x-intercept of L_1 is 3.1 and the the x-intercept of L_2 is 5.2 $\Rightarrow$ the area of the triangle = $0.5(5.2-3.1)3=3.15\ \left(\frac{63}{20} \text{ or } 3\tfrac{3}{20}\right)$

ROUND 3

1. The side of the regular hexagon is $2\sqrt{3} \Rightarrow$ longest diagonal is $4\sqrt{3} \Rightarrow$ area of the equilateral triangle = $\dfrac{\left(4\sqrt{3}\right)^2\sqrt{3}}{4} = 12\sqrt{3}$

2. The triangle is right. $\Rightarrow$ its area = $\dfrac{1}{2}\cdot 8\cdot 15 = 60$;

$60 = \dfrac{1}{2}2\cdot 17 + \dfrac{1}{2}4\cdot 8 + \dfrac{1}{2}15x$ **Þ** $x = 3.6\left(\dfrac{18}{5}\ or\ 3\tfrac{3}{5}\right)$



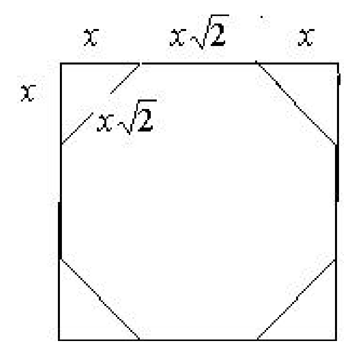
3. The side of the square = $2x + x\sqrt{2} = x\left(2+\sqrt{2}\right)$;

The area of the square = $x^2\left(2+\sqrt{2}\right)^2 = x^2\left(6+4\sqrt{2}\right)$;

The area of the 4 rt. iso. Δ's = $2x^2$; therefore the area of octagon = $x^2\left(6+4\sqrt{2}\right) - 2x^2 = x^2\left(4+4\sqrt{2}\right) = 4 \Rightarrow$

$x^2 = \dfrac{4}{4+4\sqrt{2}} = \dfrac{1}{1+\sqrt{2}} = \sqrt{2}-1 \Rightarrow$ area of the square =

$\left(\sqrt{2}-1\right)\left(4\sqrt{2}+6\right) = 2+2\sqrt{2}$



ROUND 4

1. $\left(3+\sqrt{2}\right)^2 - \sqrt[4]{4} + \dfrac{2}{\sqrt{2}-1} = 11+6\sqrt{2}-\sqrt{2}+2\left(\sqrt{2}+1\right) = 11+6\sqrt{2}-\sqrt{2}+2\sqrt{2}+2 =$
$13+7\sqrt{2} \Rightarrow x+y = 20$

2. $\log_7\left(x^3-27\right) - \log_7\left(x-3\right) = 2 \Rightarrow \log_7\left(\dfrac{x^3-27}{x-3}\right) = 2 \Rightarrow x^2+3x+9 = 49 \Rightarrow$
$x^2+3x-40 = 0 \Rightarrow (x+8)(x-5) = 0 \Rightarrow x = 5$ Note x = -8 is extraneous to the original equation.

3. $\log_{a^2} b + \log_{a^3} 2b = \log_a \sqrt[6]{x} \Rightarrow \dfrac{\log b}{\log a^2} + \dfrac{\log 2b}{\log a^3} = \dfrac{\frac{1}{6}\log x}{\log a} \Rightarrow \dfrac{\log b}{2\log a} + \dfrac{\log 2b}{3\log a} = \dfrac{\frac{1}{6}\log x}{\log a} \Rightarrow$
$3\log b + 2\log 2b = \log x \Rightarrow \log\left(b^3(2b)^2\right) = \log x \Rightarrow x = 4b^5$

ROUND 5

1. $\sin q = \frac{3}{4}$ and $\cos q < 0 \Rightarrow \cos q = -\sqrt{1-\left(\frac{3}{4}\right)^2} = -\frac{\sqrt{7}}{4}$;

$\tan\left(90^\circ + q\right) = -\cot q = -\frac{\cos q}{\sin q} = -\frac{-\frac{\sqrt{7}}{4}}{\frac{3}{4}} = \frac{\sqrt{7}}{3}$

2. $\cos q = 2\sin 19.5^\circ \cos 19.5^\circ \Rightarrow \cos q = \sin 39^\circ = \cos 51^\circ = \cos\left(360^\circ - 51^\circ\right) \Rightarrow q = 51^\circ, 309^\circ$

3. $z^3 = 13.5 - 13.5i\sqrt{3} \Rightarrow z^3 = 13.5\left(1 - i\sqrt{3}\right) = 13.5\left(2\,cis\,300^\circ\right) = 27cis300^\circ \Rightarrow$ since z is in quadrant II, then $z = 3cis100^\circ$;

$-6\sqrt{2} - 6i\sqrt{2} = 6\left(-\sqrt{2} - i\sqrt{2}\right) = 6\left(2\,cis225^\circ\right) = 12\,cis225^\circ$

$w = \frac{12\,cis\,225^\circ}{3cis100^\circ} = 4\,cis125^\circ$

TEAM ROUND

1. Since $\log_3 12 - \log_2 18 = a$ and $\log_2 3 = b$,

$\frac{2\log 2 + \log 3}{\log 3} - \frac{2\log 3 + \log 2}{\log 2} = a \Rightarrow a = \frac{2\log 2}{\log 3} - \frac{2\log 3}{\log 2} = 2\left(\frac{\log 2}{\log 3}\right) - 2\left(\frac{\log 3}{\log 2}\right) \Rightarrow$

$a = \frac{2}{b} - 2b\left(\frac{2-2b^2}{b}\ or\ \frac{2\left(1-b^2\right)}{b}\right)$

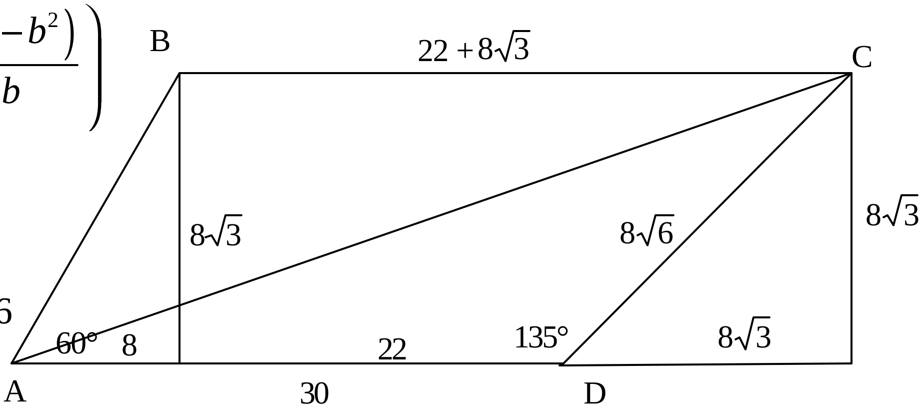
2. BC = $22 + 8\sqrt{3}$

height of Δ ABC = $8\sqrt{3}$

area of Δ ABC =

$4\sqrt{3}\left(22 + 8\sqrt{3}\right) = 88\sqrt{3} + 96$

or $8\left(11\sqrt{3} + 12\right)$



3. $2x - y + 9 = 0$ and $x + 3y - 6 = 0 \Rightarrow 6x - 3y + 27 = 0 \Rightarrow 7x + 21 = 0 \Rightarrow x = -3\ and\ y = 3$;

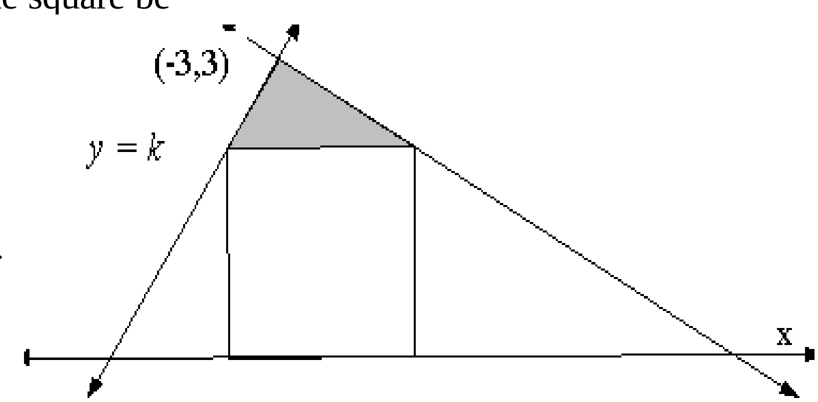
let the equation of the top side of the square be

$y = k$: $x = \frac{k-9}{2}$ and on the other

line: $x = 6 - 3k$; $6 - 3k - \frac{k-9}{2} = k$

$\Rightarrow 12 - 6k - k + 9 = 2k \Rightarrow k = \frac{7}{3} \Rightarrow$

area of triangle = $\frac{1}{2}\cdot\frac{7}{3}\left(3 - \frac{7}{3}\right) = \frac{7}{9}$



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

ANSWER SHEET:

ROUND 1

1. 4
2. $\frac{45}{4}$ $(11.25 \text{ or } 11\frac{1}{4})$
3. $-42, -50$

ROUND 2

1. (6, - 4)
2. 16 (0, 16) is acceptable.
3. 3.15 $\left(\frac{63}{20} \text{ or } 3\frac{3}{20}\right)$

ROUND 3

1. $12\sqrt{3}$
2. $3.6\left(\frac{18}{5} \text{ or } 3\frac{3}{5}\right)$
3. $2+2\sqrt{2}$

ROUND 4

1. 20
2. 5
3. $4b^5$

ROUND 5

1. $\frac{\sqrt{7}}{3}$
2. $51°, 309°$
3. $4\ cis 125°$

TEAM ROUND

3 pts. 1. $\frac{2}{b}-2b\left(\frac{2-2b^2}{b} \text{ or } \frac{2(1-b^2)}{b}\right)$

3 pts. 2. $88\sqrt{3}+96$ $\left(8\left(11\sqrt{3}+12\right)\right)$

4 pts. 3. $\frac{7}{9}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

ROUND 1 – Algebra 1: Fractions and Word Problems

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Alex takes twice as long as Mary to change a tire. Together they can change a tire in 12 minutes. How many minutes would it take Alex to change a tire working alone?

2. A motor boat travels a certain distance, d, upstream against a 4 kilometers per hour current and then returns downstream to its starting point. Traveling that same total distance, $2d$, would have taken 50% more time on a lake without any current. What is the exact number of kilometers per hour for the speed of the motorboat without any current?

3. Find all values for x satisfying the following equation: $\dfrac{x^4 - x^2 - 2x - 1}{x^3 - 1} = \dfrac{5}{2}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

ROUND 2 – Coordinate Geometry of the Straight Line

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given line $\ell, \{(x,y) | 2x - 3y = 12\}$, find the area of the triangle formed by line ℓ, the line $x = 3$ and the line $y = 2$.

2. Given points $A(-4,5), B(2,-7)$, and P, on $\overline{AB}$ such that $AP : PB = 1 : 2$. Line L is drawn through point P perpendicular to $\overline{AB}$. Find the x-intercept of line L.

3. Given line $L_1, \{(x, y) | 3x - 4y = 24\}$ and point $P(9, -8)$, line L_2, with negative slope, is drawn through point P making a 45° angle with the x axis. Find the area of the quadrilateral formed by lines L_1, L_2, and the coordinate axes.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

ROUND 3 – Geometry: Polygons: Area and Perimeter

1. ______________________

2. ______________________

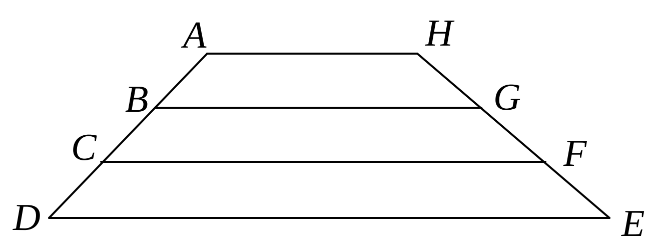
3. ______________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the ratio of the lengths of the diagonals of a rhombus is 2:3 and its area is 48 square inches, find the number of inches in the perimeter of this rhombus.

2. Find the number of square centimeters in the area of a right triangle with hypotenuse of length 10 cm and the lengths of its legs are in the ratio of 1:3.

3. Segments $\overline{AH}, \overline{BG}, \overline{CF}$, and $\overline{DE}$ are parallel, $\overline{EFGH}$, with points B and C trisecting $\overline{AD}$. If $AH = 3$ and $DE = 7$, find the ratio of the area of trapezoid $ABGH$ to the area of trapezoid $ADEH$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

ROUND 4 – Algebra 2– Logs, Exponents, Radicals and equations involving them

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all positive values for *x* satisfying the following equation: $\left(\sqrt[10]{x}\right)\left(\sqrt[5]{x}\right)=\left(\sqrt{7}\right)\left(\sqrt[20]{x}\right)$

2. Find all values for *x* satisfying the following equation: $\log_8 \sqrt{2} = \log_x \sqrt{3} - \log_x \sqrt[3]{9}$

3. Find all values for *x* satisfying the following equation: $\log_9 x = \log_{25} 125 - \log_x 3$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

ROUND 5 – Trig. Analysis and Complex Numbers, Trig Form

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all complex solutions to the equation $z^3 = -4 + 4i\sqrt{3}$ in the polar form $r \operatorname{cis} q$, where $r > 0$ and $0° \le q < 360°$.

2. Find all values of x such that $0° \le x < 360°$ and

   $\left(\sqrt{2}\operatorname{cis}315°\right)^6 = \left(4\operatorname{cis}855°\right)^2 \cos\left(270° + x\right)$

3. Find all solutions to the equation $\sin\left(x + 40°\right) + \sin\left(x - 40°\right) = \sin 50° \cdot \tan x$ where $0° \le x < 360°$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

TEAM ROUND

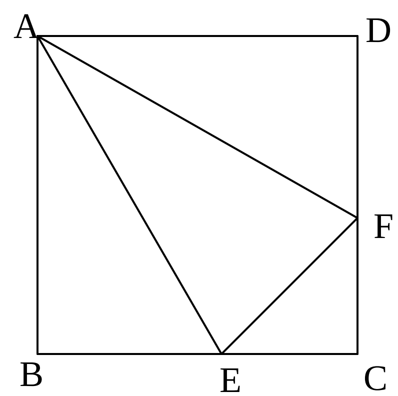
3 pts. 1. ____________________

3 pts. 2. ____________________

4 pts. 3. ____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given square ABCD and isosceles ΔAEF with base $\overline{EF}$ such that $m\angle EAF = 30°$ as indicated on the diagram on the right. If CE = 2, find the exact area of ΔAEF.



2. If $\log_6 12 = k$, find $\log_2 3$ as a simplified expression in terms of *k*.

3. Given $0° < x < 45°$, $a > 4$ and $\tan x + \cot x = \sqrt{a}$, find in <u>simplest radical form</u> the value for $\cos 2x$ in terms of a.

Detailed Solutions of GBML for MEET 3 – DECEMBER 2000

ROUND 1

1. Let t = number of minutes for Mary $\rightarrow$ $2t$ = minutes for Alex $\rightarrow$ $\frac{12}{t}+\frac{12}{2t}=1\rightarrow 2t=36$

2. Let d = distance one way, r = rate of the boat in still water $\rightarrow$ $\frac{d}{r+4}+\frac{d}{r-4}=\frac{3}{2}\cdot\frac{2d}{r}\rightarrow$
$\frac{1}{r+4}+\frac{1}{r-4}=\frac{3}{r}\rightarrow r(r-4)+r(r+4)=3\left(r^2-16\right)\rightarrow$
$r^2+4r+r^2-4r=3r^2-48\rightarrow r^2=48\rightarrow r=4\sqrt{3}$

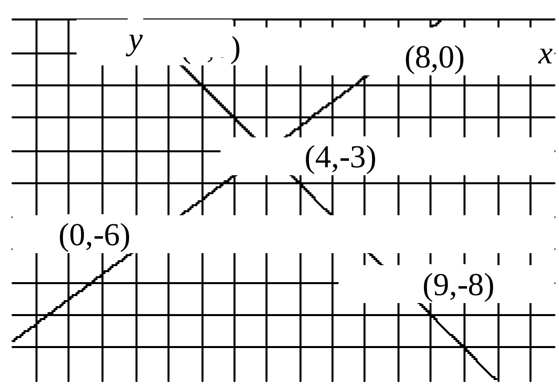
3. $\frac{x^4-x^2-2x-1}{x^3-1}=\frac{5}{2}\rightarrow\frac{x^4-(x+1)^2}{(x-1)\left(x^2+x+1\right)}=\frac{5}{2}\rightarrow\frac{\left(x^2-x-1\right)\left(x^2+x+1\right)}{(x-1)\left(x^2+x+1\right)}=\frac{5}{2}\rightarrow$
$\frac{x^2-x-1}{x-1}=\frac{5}{2}\rightarrow 2x^2-2x-2=5x-5\rightarrow 2x^2-7x+3=0\rightarrow$
$(2x-1)(x-3)=0\rightarrow x=\frac{1}{2},3$

ROUND 2

1. When $x = 3$: $6-3y=12\rightarrow y=-2$; when $y=2: 2x-6=12\rightarrow x=9\rightarrow$ vertices of the triangle are $A(3,2)$, $B(3,-2)$, and $C(9,2)$ $\rightarrow$ area of ΔABC = $\frac{1}{2}\cdot 4\cdot 6=12$

2. slope of line $L\perp$ to $\overline{AB}=-\left(\frac{-7-5}{2+4}\right)^{-1}=\frac{1}{2}$; $P=\left(\frac{2(-4)+1(2)}{1+2},\frac{2(5)+1(-7)}{1+2}\right)=(-2,1)$;
line L: $y-1=\frac{1}{2}(x+2)\rightarrow y=\frac{1}{2}x+2\rightarrow(-4,0)$ on L.

3. line $L_2: y+8=-1(x-9)\rightarrow y=-x+1\rightarrow(1,0)$ is its x-intercept; $3x-4(-x+1)=24\rightarrow$ $7x=28\rightarrow(4,-3)$ is the point of intersection of L_1 and L_2; area of the quadrilateral = $\frac{1}{2}\cdot 6\cdot 8-\frac{1}{2}\cdot 7\cdot 3=\frac{27}{2}$

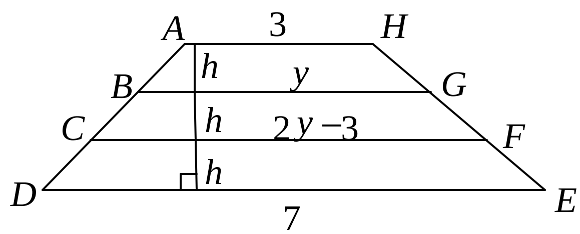


ROUND 3

1. Let the diagonal be $2x$ and $3x$ inches long. $\frac{1}{2}(2x)(3x)=48\rightarrow 3x^2=48\rightarrow x=4$;
the length of one side = $\sqrt{4^2+6^2}=2\sqrt{13}\rightarrow$ Perimeter $=8\sqrt{13}$ inches.

2. $x^2+(3x)^2=10^2\rightarrow 10x^2=100\rightarrow x=\sqrt{10}\rightarrow \text{area}=\frac{1}{2}\left(\sqrt{10}\right)\left(3\sqrt{10}\right)=15\text{cm}^2$

3. Because of the median property of trapezoids, if $BG = y$, $CF = 2y - 3$
and $7+y=4y-6\rightarrow y=\frac{13}{3}$.
The ratio of areas = $\dfrac{\frac{1}{2}\left(\frac{22}{3}\right)(h)}{\frac{1}{2}(10)(3h)}=11:45$



ROUND 4

1. $\left(\sqrt[10]{x}\right)\left(\sqrt[5]{x}\right)=\left(\sqrt{7}\right)\left(\sqrt[20]{x}\right)\rightarrow \dfrac{x^{\frac{1}{5}}x^{\frac{1}{10}}}{x^{\frac{1}{20}}}=7^{\frac{1}{2}}\rightarrow x^{\frac{1}{4}}=7^{\frac{1}{2}}\rightarrow x=7^2\rightarrow x=49$

2. $\log_8\sqrt{2}=\log_x\sqrt{3}-\log_x\sqrt[3]{9}\rightarrow \dfrac{\log\sqrt{2}}{\log 8}=\frac{1}{2}\log_x 3-\frac{2}{3}\log_x 3\rightarrow$
$\dfrac{\frac{1}{2}\log 2}{3\log 2}=-\frac{1}{6}\log_x 3\rightarrow \frac{1}{6}=-\frac{1}{6}\log_x 3\rightarrow \log_x 3=-1\rightarrow x=\frac{1}{3}$

3. $\log_9 x=\log_{25}125-\log_x 3\rightarrow \frac{1}{2}\log_3 x=\frac{3}{2}-\dfrac{1}{\log_3 x}$;
Let $y=\log_3 x\rightarrow \frac{1}{2}y=\frac{3}{2}-\frac{1}{y}\rightarrow y^2=3y-2\rightarrow y^2-3y+2=0\rightarrow$
$(y-1)(y-2)=0\rightarrow \log_3 x=1,2\rightarrow x=3,9$

ROUND 5

1. $z^3 = -4 + 4i\sqrt{3} = 4\left(-1 + i\sqrt{3}\right) = 4\left(2\text{cis}120°\right) = 8\text{cis}120°$

$\rightarrow z = \sqrt[3]{8}\text{cis}\left(\dfrac{120°}{3} + \dfrac{360°}{3}n\right), n = 0,1,2 \rightarrow z = 2\text{cis}40°, 2\text{cis}160°, 2\text{cis}280°$

2. $\left(\sqrt{2}\text{cis}315\right)^6 = \left(4\text{cis}855°\right)^2 \cos\left(270° + x\right) \rightarrow$

$\rightarrow 8\text{cis}1890° = 16\text{cis}1710° \sin x \rightarrow$

$\sin x = \dfrac{1}{2}\text{cis}180° = -\dfrac{1}{2} \rightarrow x = 210°, 330°$

3. $\sin\left(x + 40°\right) + \sin\left(x - 40°\right) = \sin 50° \cdot \tan x \rightarrow$

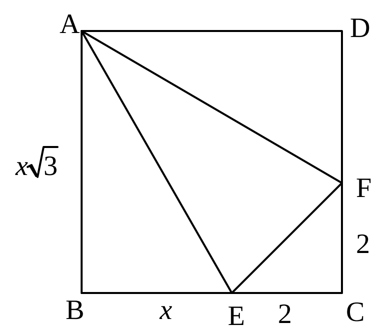
$\sin x \cos 40° + \cos x \sin 40° + \sin x \cos 40° - \cos x \sin 40° = \sin 50° \cdot \tan x \rightarrow$

$2\sin x \cos 40° = \cos 40° \cdot \dfrac{\sin x}{\cos x} \rightarrow 2\sin x = \dfrac{\sin x}{\cos x} \rightarrow 2\sin x \cos x - \sin x = 0$ and $\cos x \neq 0 \rightarrow$

$\sin x\left(2\cos x - 1\right) = 0$ and $\cos x \neq 0 \rightarrow \sin x = 0, \cos x = \dfrac{1}{2}$, and $\cos x \neq 0 \rightarrow$

$x = 0°, 60°, 180°, 300°$

TEAM ROUND



1. ΔABE is a 30-60-90° triangle; BE = x, AB = $x\sqrt{3}$;

$x + 2 = x\sqrt{3} \rightarrow x = \dfrac{2}{\sqrt{3} - 1} = \sqrt{3} + 1$; area of ΔAEF =

$\left(x + 2\right)^2 - x^2\sqrt{3} - 2 = \left(3 + \sqrt{3}\right)^2 - \left(1 + \sqrt{3}\right)^2 \sqrt{3} - 2 =$

$12 + 6\sqrt{3} - \sqrt{3}\left(4 + 2\sqrt{3}\right) - 2 = 4 + 2\sqrt{3}$

2. $\log_6 12 = k \rightarrow \dfrac{\log_2 12}{\log_2 6} = k \rightarrow \dfrac{\log_2 4 + \log_2 3}{\log_2 2 + \log_2 3} = k \rightarrow \dfrac{2 + \log_2 3}{1 + \log_2 3} = k \rightarrow$

$2 + \log_2 3 = k + k\log_2 3 \rightarrow \log_2 3\left(1 - k\right) = k - 2 \rightarrow \log_2 3 = \dfrac{k - 2}{1 - k}$

3. $0° < x < 45°$ and $\tan x + \cot x = \sqrt{a} \rightarrow \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = \sqrt{a} \rightarrow \dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \sqrt{a} \rightarrow$

$2\sin x \cos x = \dfrac{2}{\sqrt{a}} \rightarrow \sin 2x = \dfrac{2}{\sqrt{a}}$ and

$0° < 2x < 90° \rightarrow \cos 2x = \sqrt{1 - \sin^2 2x} = \sqrt{1 - \dfrac{4}{a}} = \dfrac{\sqrt{a^2 - 4a}}{a}$;

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

ANSWER SHEET:

ROUND 1

1. 36 (36 minutes)
2. $4\sqrt{3}$ $\left(4\sqrt{3}\,kph\right)$
3. $\frac{1}{2}, 3$

ROUND 2

1. 12
2. –4 (or $(-4, 0)$)
3. $\frac{27}{2}$ (or 13.5 or 13½)

ROUND 3

1. $8\sqrt{13}$ $\left(8\sqrt{13}\text{inches}\right)$
2. 15 (15cm^2)
3. 11:45 $\left(\frac{11}{45}\right)$

ROUND 4

1. 49
2. $\frac{1}{3}$
3. 3, 9

ROUND 5

1. $2\text{cis}40°, 2\text{cis}160°, 2\text{cis}280°$
2. 210°, 330°
3. 0°, 60°, 180°, 300°

TEAM ROUND

3 pts. 1. $4+2\sqrt{3}$

3 pts. 2. $\frac{k-2}{1-k}$ $\left(\text{or } \frac{2-k}{k-1}\right)$

4 pts. 3. $\frac{\sqrt{a^2-4a}}{a}$ $\left(\text{or } \frac{\sqrt{a(a-4)}}{a}\right)$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

ROUND 1 – Algebra 1: Fractions and Word Problems

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The ratio of girls to boys at a math meet is 2:3. If two more girls and eight more boys compete, then the ratio of girls to boys would be 5:8. Find the total number of students at the meet originally.

2. Solve the following equation for x:

$$\frac{x-1-\frac{1}{x-1}}{x-3+\frac{2}{x}}=1$$

3. Jill, a master carpenter, and two trainees, Jack and Jim, are building a room to a house. If each one worked alone, Jack would take six hours longer to build the room than Jill and Jim takes a third longer than Jack. Jack and Jill without Jim worked for six hours and then stopped. Jim by himself finished building the room in twenty-two hours. How many hours would it have taken Jill to build the room from start to finish without any help?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

ROUND 2 – Coordinate Geometry of the Straight Line

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A triangle is formed by the intersection of line $\ell : \{(x, y) | 2x - 3y - 36 = 0\}$ and the coordinate axes. The line $y = mx$ divides this triangle into two triangles with equal area. Find the value for m.

2. Given points $P(-5,-1)$ and $Q(7,14)$, point R is on $\overline{PQ}$ such that $PR : RQ = 1 : 2$, and S is a point on the x axis such that $\overline{RS}$ is perpendicular to $\overline{PQ}$, find the first coordinate of point S.

3. Given point $P(-2a, a+4)$ lies on line L, $\{(x, y) | 3ax + 7y = 5a\}$, solve for a.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

ROUND 3 – Geometry: Polygons: Area and Perimeter

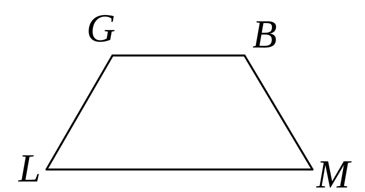
1. ____________________

2. ____________________

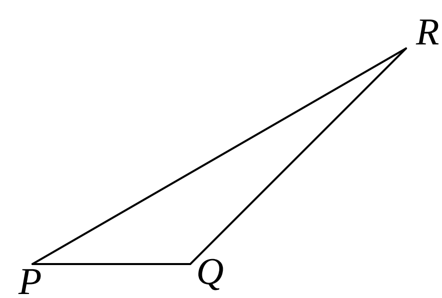
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

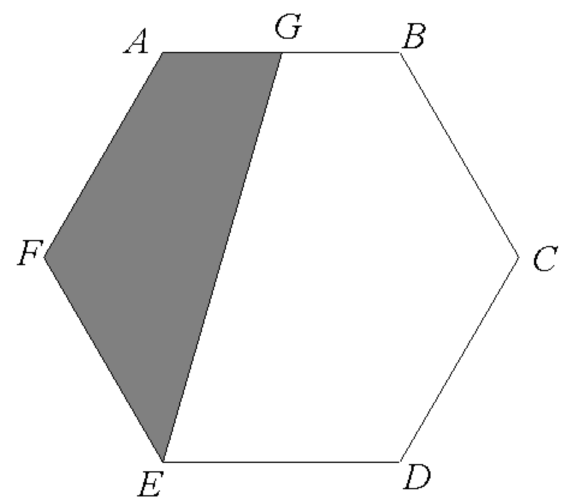
1. Given $LG = GB = BM$, $m\angle G = m\angle B = 120°$, and the area of quadrilateral $GBML = 48\sqrt{3}$, find its perimeter.



2. Given $m\angle P = 30°$, $m\angle Q = 135°$, and $QR = 2$, find the area of ΔPQR.



3. Given *ABCDEF* is a regular hexagon and *G* is the midpoint of $\overline{AB}$, find the ratio of the shaded area *AFEG* to the unshaded area *GBCDE*.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

ROUND 4 – Algebra 2– Logs, Exponents, Radicals and equations involving them

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Solve for x if $\dfrac{x}{\sqrt{x}\cdot\sqrt[6]{x}}=\sqrt[4]{5x}$.

2. Solve the following equation for x: $\log_6 x+\log_6(x-1)=\log_6(x-2)+1$.

3. Given $\dfrac{\log_3 4-\log_{27}4}{\log_{\sqrt[3]{5}}2+\log_{125}\sqrt{2}-\log_{25}32}=\log_b a$ where a and b are integers, find the least possible value for $a+b$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

ROUND 5 – Trig. Analysis and Complex Numbers, Trig Form

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find $\left(\frac{1}{2}cis\,20°\right)^5 \left(2cis\,26°\right)^{10}$ in rectangular form.

2. Given $\tan a = \frac{cis\,270°}{2cis\,90°}$ and $\cos a < 0$, find the value for $\sin(2a)$.

3. Given $0 \le x \le 180$, $0 \le y \le 180$, $\cos x° \cos y° - \sin x° \sin y° = -0.5$, and $\sin x° \cos y° - \cos x° \sin y° = 1$, find all possible ordered pairs (x, y).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

TEAM ROUND (<u>12 MINUTES LONG</u>)

3 pts. 1. ___________________

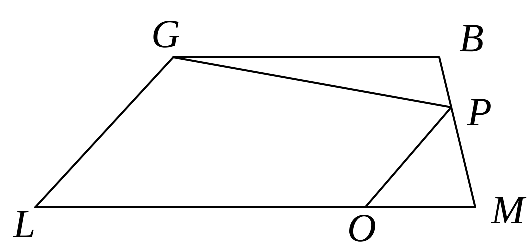
3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given $L_1:\{(x,y)\,|\,ax-4y=-11\}$ and $L_2:\{(x,y)\,|\,5x+6y=-4a\}$ intersect at point $P(-a,b)$, find all possible values for a.

2. Given $\overline{GB}\,||\,\overline{ML}$, $\overline{BPM}$, $\overline{LOM}$, $GB:ML=3:5$, $BP:PM=1:2$, $LO:OM=4:1$, and the area of quadrilateral $GLOP$ = 114, find the area of trapezoid $GBML$.



3. Given $a>0$, find in terms of a the area of region $\Re$, $\{(x,y)\,|\,y\ge|2x-4a|+a \text{ and } y\le x+2a\}$.

Detailed solutions to GBML Meet 3 2001

ROUND 1 – Algebra 1: Fractions and Word Problems

1. Let 2*x* = number of girls and 3*x* = number of girls ⇒ 5*x* = total number of students ⇒ $\frac{2x+2}{3x+8}=\frac{5}{8}\Rightarrow 16x+16=15x+40\Rightarrow x=24\Rightarrow 5x=120.$

2. $\dfrac{x-1-\frac{1}{x-1}}{x-3+\frac{2}{x}}=1 \Rightarrow \dfrac{\frac{(x-1)^2}{x-1}-\frac{1}{x-1}}{\frac{x^2-3x+2}{x}}=1\Rightarrow \dfrac{\frac{x^2-2x}{x-1}}{\frac{(x-2)(x-1)}{x}}=1\Rightarrow$

$\frac{x\cancel{(x-2)}}{x-1}\cdot\frac{x}{(x-1)\cancel{(x-2)}}=1\Rightarrow \frac{x^2}{(x-1)^2}=1\Rightarrow \frac{x}{x-1}=\pm1\Rightarrow x=x-1\; or\; x=1-x\Rightarrow x=\frac{1}{2}.$

3. Let *x* = hours for Jill to do the job ⇒ $x+6=$ hours for Jack and $\frac{4}{3}(x+6)=$ hours for Jim

$\Rightarrow \frac{6}{x}+\frac{6}{x+6}+\frac{22}{\frac{4}{3}(x+6)}=1\Rightarrow \frac{6}{x}+\frac{6}{x+6}+\frac{33}{2(x+6)}=1\Rightarrow \frac{6}{x}+\frac{45}{2(x+6)}=1$

$\Rightarrow 12(x+6)+45x=2x(x+6)\Rightarrow 12x+72+45x=2x^2+12x\Rightarrow\; 0=2x^2-45x-72\;\Rightarrow$
$(2x+3)(x-24)=0\Rightarrow x=24$ hours.

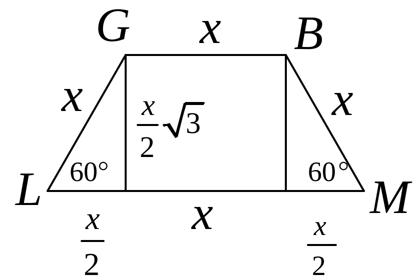
ROUND 2 – Coordinate Geometry of the Straight Line

1. The line ℓ intersects the axes at points $P\ (18,0)$ and $Q\ (0,-12)$. For the line $y=mx$ to divide the triangle into two triangles with equal area the line would pass through the midpoint of $\overline{PQ}$ which is $(9,-6)\Rightarrow m=\frac{-6}{9}=-\frac{2}{3}$.

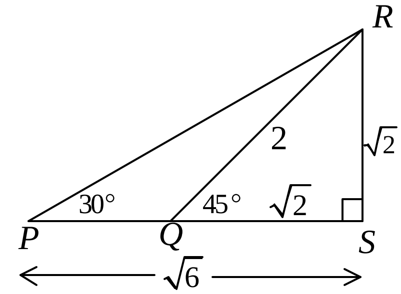
2. To find the coordinates of S: $x=\frac{2(-5)+1(7)}{1+2}=-1$ and $y=\frac{2(-1)+1(14)}{1+2}=4$. The slope of $\overline{PQ}=\frac{14+1}{7+5}=\frac{15}{12}=\frac{5}{4}\Rightarrow$ slope of $\overline{RS}=-\frac{4}{5}\Rightarrow$ equation of line $\overleftrightarrow{RS}$ is $y-4=-\frac{4}{5}(x+1)\Rightarrow$ If $y=0\Rightarrow -4=-\frac{4}{5}(x+1)\Rightarrow 5=x+1\Rightarrow x=4$.

3. Substituting the coordinates of P into the equation of L: $3a(-2a)+7(a+4)=5a\Rightarrow$ $-6a^2+7a+28=5a\Rightarrow 6a^2-2a-28=0\Rightarrow 3a^2-a-14=0\Rightarrow$ $(3a-7)(a+2)=0\Rightarrow a=-2,\frac{7}{3}$

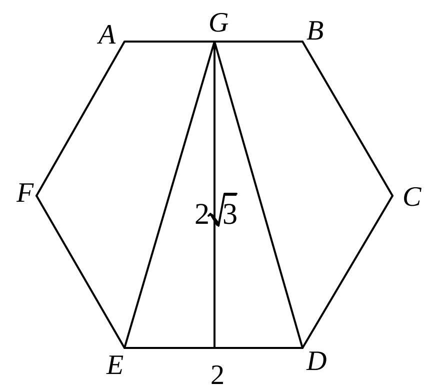
ROUND 3 – Geometry: Polygons: Area and Perimeter



1. Let $LG = x \Rightarrow$ perimeter of $GBML = 5x$. The area of $GBML = x\cdot\frac{x}{2}\sqrt{3} + 2\cdot\frac{1}{2}\cdot\frac{x}{2}\cdot\frac{x}{2}\sqrt{3} = \frac{3}{4}x^2\sqrt{3} = 48\sqrt{3} \Rightarrow$ $x^2 = 64 \Rightarrow x = 8 \Rightarrow 5x = 40$.



2. Extend $\overline{PQ}$ until it meets the altitude $\overline{RS}$. Since $RQ = 2$ $\Rightarrow RS = QS = \sqrt{2} \Rightarrow PS = \sqrt{6} \Rightarrow PQ = \sqrt{6} - \sqrt{2}$. The area of $\Delta PQR = \frac{1}{2}\left(\sqrt{6}-\sqrt{2}\right)\sqrt{2} = \frac{2\sqrt{3}-2}{2} = \sqrt{3}-1$.



3. Let the side of the hexagon = 2. Draw $\overline{GD}$ and a perpendicular from G to $\overline{DE}$. The length of this perpendicular = $2\sqrt{3}$ (same length as $\overline{AE}$) $\Rightarrow$ area of ΔDEG = $2\sqrt{3}$. The area of the hexagon = $6\cdot\frac{2^2\sqrt{3}}{4} = 6\sqrt{3}$. Since $AGEF \cong BGDC \Rightarrow$ area of $AGEF = 2\sqrt{3}$ $\Rightarrow$ ratio of area of $AGEF$: area of $GBCDE$ = 1:2.

ROUND 4 – Algebra 2– Logs, Exponents, Radicals and equations involving them

1. $\frac{x}{\sqrt{x}\cdot\sqrt[6]{x}} = \sqrt[4]{5x} \Rightarrow \frac{x}{x^{1/2}x^{1/6}} = 5^{1/4}x^{1/4} \Rightarrow x^{1-1/2-1/6-1/4} = 5^{1/4} \Rightarrow x^{1/12} = 5^{1/4} \Rightarrow x = \left(5^{1/4}\right)^{12} = 5^3 = 125$.

2. $\log_6 x + \log_6(x-1) = \log_6(x-2) + 1 \Rightarrow \log_6 x + \log_6(x-1) = \log_6(x-2) + \log_6 6 \Rightarrow$
$\log_6 x(x-1) = \log_6 6(x-2) \Rightarrow x^2 - x = 6x - 12 \Rightarrow x^2 - 7x + 12 = 0 \Rightarrow$
$(x-3)(x-4) = 0 \Rightarrow x = 3,4$. (Both solutions satisfy the original equation.)

3. $$\frac{\log_3 4 - \log_{27} 4}{\log_{\sqrt[3]{5}} 2 + \log_{125}\sqrt{2} - \log_{25} 32} = \frac{\frac{2\log 2}{\log 3} - \frac{2\log 2}{3\log 3}}{\frac{\log 2}{\frac{1}{3}\log 5} + \frac{\frac{1}{2}\log 2}{3\log 5} - \frac{5\log 2}{2\log 5}} = \frac{\left(2-\frac{2}{3}\right)\frac{\log 2}{\log 3}}{\frac{3\log 2}{\log 5} + \frac{\log 2}{6\log 5} - \frac{5\log 2}{2\log 5}} =$$
$$\frac{\frac{4\log 2}{3\log 3}}{\left(3+\frac{1}{6}-\frac{5}{2}\right)\frac{\log 2}{\log 5}} = \frac{\frac{4\log 2}{3\log 3}}{\frac{2\log 2}{3\log 5}} = \frac{2\log 5}{\log 3} = \frac{\log 25}{\log 3} = \log_3 25 \Rightarrow a + b = 28.$$

ROUND 5 – Trig. Analysis and Complex Numbers, Trig Form

1. $\left(\frac{1}{2}cis\,20°\right)^5 (2cis\,26°)^{10} = \left(2^{-5}cis100°\right)\left(2^{10}cis\,260°\right) = 2^5 cis360° = 32$ or $32+0i$

2. $\tan a = \frac{cis\,270°}{2cis\,90°} = \frac{1}{2}cis180° = -\frac{1}{2}$. Since $\cos a < 0 \Rightarrow \sin a > 0 \Rightarrow y = 1 \; and \; x = -2 \Rightarrow$

$r = \sqrt{5} \Rightarrow \sin a = \frac{1}{\sqrt{5}}, \cos a = -\frac{2}{\sqrt{5}} \Rightarrow \sin 2a = 2\sin a \cos a = 2\left(\frac{1}{\sqrt{5}}\right)\left(-\frac{2}{\sqrt{5}}\right) = -\frac{4}{5}$.

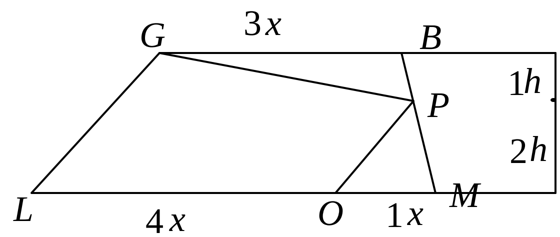
3. Given $0 \le x \le 180$, $0 \le y \le 180$, $\cos x° \cos y° - \sin x° \sin y° = -0.5$, and $\sin x° \cos y° - \cos x° \sin y° = 1 \Rightarrow \cos(x+y) = -0.5$ and $\sin(x-y) = 1 \Rightarrow$ $x+y = 120$ or 240 and $x - y = 90 \Rightarrow 2x = 210$ or $330 \Rightarrow x = 105$ or $165 \Rightarrow$ $y = 15$ or 75 respectively $\Rightarrow$ ordered pairs are $(105,15),(165,75)$.

TEAM ROUND

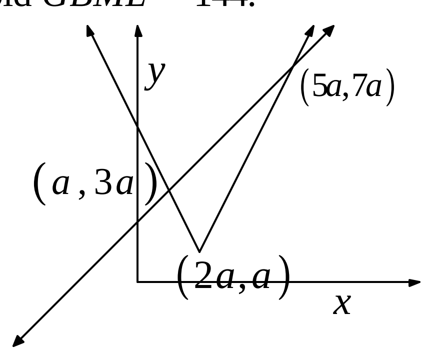
1. Given $L_1 : \{(x,y) \mid ax - 4y = -11\}$ and $L_2 : \{(x,y) \mid 5x + 6y = -4a\}$ intersect at point $P(-a,b) \Rightarrow 3ax - 12y = -33$ and $10x + 12y = -8a \Rightarrow 3ax + 10x = -33 - 8a \Rightarrow$

$x = \frac{-33-8a}{3a+10} = -a \Rightarrow -33 - 8a = -3a^2 - 10a \Rightarrow 3a^2 + 2a - 33 = 0 \Rightarrow (3a+11)(a-3) = 0$

$\Rightarrow a = -\frac{11}{3}, 3.$



2. Let $GB = 3x \Rightarrow ML = 5x \Rightarrow LO = 4x$ and $OM = x$
Let h = distance from P to $\overleftrightarrow{GB} \Rightarrow 2h$ = distance from P to $\overleftrightarrow{LM} \Rightarrow$ distance between $\overline{GB}$ and $\overline{ML} = 3h$. The area of trapezoid $GBML = \frac{1}{2} \cdot 3h \cdot (3x + 5x) = 12hx$. The area of $\Delta GBP = \frac{1}{2}3hx = \frac{3}{2}hx$ and the area of $\Delta OMP = \frac{1}{2}x \cdot 2h = hx \Rightarrow$ area of quadrilateral $GPOL$ $= 12hx - \frac{3}{2}hx - hx = \frac{19}{2}hx = 114 \Rightarrow hx = 12 \Rightarrow$ area of trapezoid $GBML$ = 144.



3. $\{(x,y) \mid y \ge |2x - 4a| + a$ and $y \le x + 2a\}$. The vertex of the absolute value inequality = $(2a,a)$.
When $x \ge 2a \Rightarrow y = 2x - 4a + a = 2x - 3a$. When $x < 2a \Rightarrow y = 4a - 2x + a = 4 = 5a - 2x$.
$x + 2a = 2x - 3a \Rightarrow x = 5a \Rightarrow y = 7a.$
$x + 2a = 5a - 2x \Rightarrow x = a \Rightarrow y = 3a.$ The area of the triangle formed by these 3 points $= \frac{1}{2}\text{abs}\begin{vmatrix} 2a & a & 1 \\ a & 3a & 1 \\ 5a & 7a & 1 \end{vmatrix} = \frac{a^2}{2} abs \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 5 & 7 & 1 \end{vmatrix} = 6a^2$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

ANSWER SHEET:

ROUND 1

1. 120
2. $\frac{1}{2}$
3. 24 (24 hours)

ROUND 2

1. $-\frac{2}{3}$
2. 4
3. $-2, \frac{7}{3}$

ROUND 3

1. 40
2. $\sqrt{3}-1$
3. 1:2 $\left(\text{or } \frac{1}{2}\right)$

ROUND 4

1. 125
2. 3, 4
3. 28

ROUND 5

1. 32 $(\text{or } 32+0i)$
2. $-\frac{4}{5}$ $(\text{or } -0.8)$
3. $(105,15), (165,75)$

TEAM ROUND

3 pts. 1. $-\frac{11}{3}, 3$ $(\text{or } -3\tfrac{2}{3}, 3)$

3 pts. 2. 144

4 pts. 3. $6a^2$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2002

ROUND 1 – Algebra 1: Fractions and Word Problems

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Simplify the following expression: $\dfrac{1-x^{-4}}{x^{-1}+x^{-3}}+x^{-1},\ x\neq 0$

2. Solve the following equation for x: $\dfrac{3x}{2x+1}-\dfrac{x+2}{2x+3}=\dfrac{8x+1}{4x^2+8x+3}$

3. Bill takes 20% more time to type a page than Ann. Bill spent 144 minutes by himself typing a manuscript, then stopped. Ann took over and finished the manuscript in 3 hours. If the manuscript is 200 pages long, how many pages can Ann type in 6 minutes?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2002

ROUND 2 – Coordinate Geometry of the Straight Line

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given points $A(t,-6)$ and $B(8,2w)$, M is the midpoint of $\overline{AB}$ with coordinates $(3,-7)$, find the sum $t+w$.

2. Find the area of the triangle formed by the intersection of the lines whose equations are $y=2x+6$, $y=-x+6$ and $x+4y+3=0$.

3. The x and y axis cut off a segment from the line $y=-\frac{3}{2}x+3$ which forms one side of a square. Find the other two vertices of the square if they both lie in the first quadrant.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2002

ROUND 3 – Geometry: Polygons: Area and Perimeter

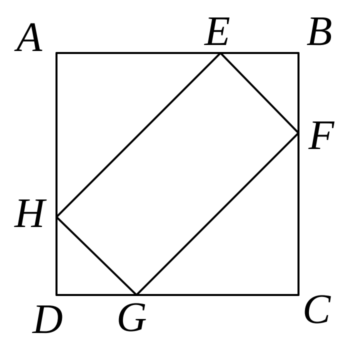
1. ____________________

2. ____________________

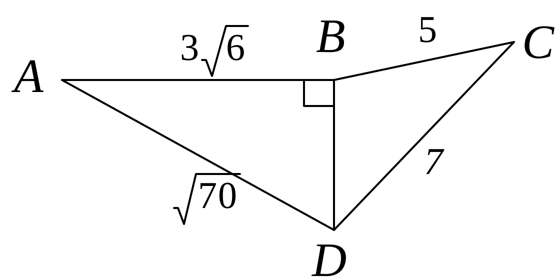
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

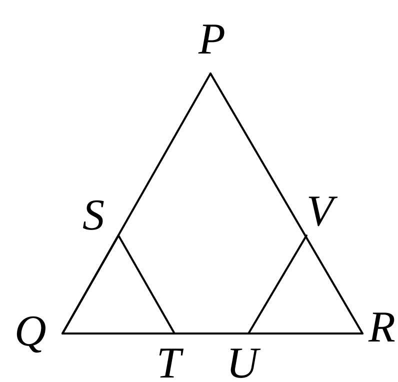
1. Given the figure on the right with square *ABCD* and rectangle *EFGH* such that $BE = BF$ and $EF : FG = 1:3$. Find the ratio of the area of the rectangle to the area of the square.



2. Given $\overline{AB} \perp \overline{BD}$ and the indicated measurements on the diagram on the right, find the area of polygon *ABCD*.



3. Given equilateral triangles *PQR* , *QST*, *RUV*. $QS = RV$, $PQ = 6$cm, and the perimeter of ΔPQR is 5 cm more than the perimeter of pentagon *PSTUV*, find the ratio of the area of pentagon *PSTUV* to the area of ΔPQR.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2002

ROUND 4 – Algebra 2– Logs, Exponents, Radicals and equations involving them

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Solve the following equation for *x*: $\log(2x+3)-\log(2x+1)=\log x$

2. Given $\log_b 9 \cdot \log_{27} 100 = 8$, find b^3 in simplest radical form.

3. Solve the following equation for *x*: $6^{2\log_6 x} - \log_3 6^x + \log_3 2^x = 132$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2002

ROUND 5 – Trig. Analysis and Complex Numbers, Trig Form

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

Note that $rcisq = r(\cos q + i\sin q)$ where $i = \sqrt{-1}$.

1. Find z in $r\,\mathrm{cis}q$ form where $r > 0$ and $0° \le q < 360°$ such that

$$z\left(1 + i\sqrt{3}\right) = -2\sqrt{2} + 2i\sqrt{2}$$

2. Given $\tan\left(x + \frac{p}{3}\right) = 4\sqrt{3}$, find the value of $\tan x$.

3. Find all possible values of z in $r\,\mathrm{cis}q$ form where $r > 0$ and $0° \le q < 360°$ such that

$$\frac{z+2}{-2i} = \frac{4+4i}{z^2 - 2z + 4}$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2002

TEAM ROUND: Time Limit: 12 Minutes

3 pts. 1. ___________________

3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. The expression $\frac{22}{2x^2+5x-12}+\frac{A}{2x^2-9x+9}$, $A\neq 0$, can be simplified into a single rational expression where the numerator is constant and the denominator is a quadratic polynomial. Find the value for A.

2. Given points $A(-6,3)$ and $B(6,11)$, with point P between A and B such that $AP:PB=3:1$. If point P is the vertex of an isosceles triangle having its base on the x axis and having one of its legs on line $\overleftrightarrow{AB}$, find the length of its base.

3. Given $y>0,\ y\neq 1,\ x>0,\ x\neq 1$, find all possibilities for y in terms of x given $\log_x y+\log_y x^9=6+\frac{1}{\log\sqrt{y}\cdot\log\sqrt{x}}$.

Detailed solutions to GBML Meet 3 2002

ROUND 1 – Algebra 1: Fractions and Word Problems

1. $\dfrac{1-x^{-4}}{x^{-1}+x^{-3}}+x^{-1}=\dfrac{x^4\left(1-x^{-4}\right)}{x^4\left(x^{-1}+x^{-3}\right)}+\dfrac{1}{x}=\dfrac{x^4-1}{x^3+x}+\dfrac{1}{x}=\dfrac{\left(x^2-1\right)\left(x^2+1\right)}{x\left(x^2+1\right)}+\dfrac{1}{x}=$

$\dfrac{x^2-1}{x}+\dfrac{1}{x}=x-\dfrac{1}{x}+\dfrac{1}{x}=x$

2. $\dfrac{3x}{2x+1}-\dfrac{x+2}{2x+3}=\dfrac{8x+1}{4x^2+8x+3}\Rightarrow\dfrac{3x}{2x+1}-\dfrac{x+2}{2x+3}=\dfrac{8x+1}{(2x+1)(2x+3)}\Rightarrow$

$3x(2x+3)-(x+2)(2x+1)=8x+1\Rightarrow 6x^2+9x-\left(2x^2+5x+2\right)=8x+1\Rightarrow$

$6x^2+9x-2x^2-5x-2=8x+1\Rightarrow 4x^2-4x-3=0\Rightarrow(2x-3)(2x+1)=0\Rightarrow$

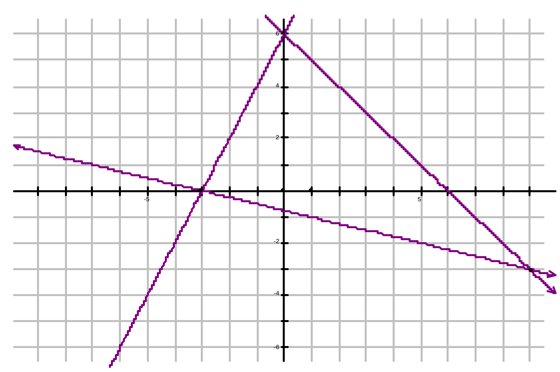
$x=\cancel{-\dfrac{1}{2}},\dfrac{3}{2}\Rightarrow x=\dfrac{3}{2};\quad x=-\dfrac{1}{2}$ makes denominators of the orginal equation equal to 0.

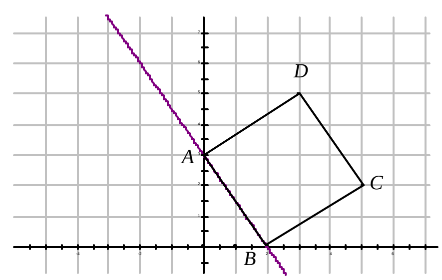
3. Let x = minutes for Ann to type the manuscript $\Rightarrow 1.2x$ = minutes for Bill to type the manuscript $\Rightarrow\dfrac{144}{1.2x}+\dfrac{180}{x}=1\Rightarrow\dfrac{120}{x}+\dfrac{180}{x}=1\Rightarrow\dfrac{300}{x}=1\Rightarrow x=300\text{min}\Rightarrow$ Ann can type 200 pages in 300 minutes $\Rightarrow\left(\frac{2}{3}\right)6=4$ pages in 6 minutes.

ROUND 2 – Coordinate Geometry of the Straight Line

1. $\dfrac{t+8}{2}=3$ and $\dfrac{-6+2w}{2}=-7\Rightarrow t+8=6$ and $-3+w=-7\Rightarrow t=-2$ and $w=-4\Rightarrow$ $t+w=-6$

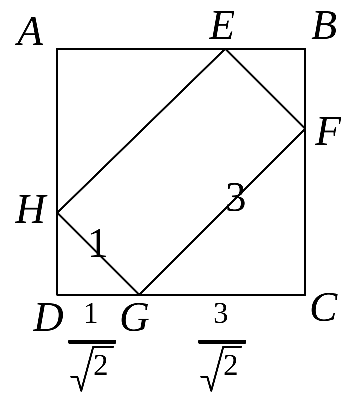
2. Given $y=2x+6$, $y=-x+6$ and $x+4y+3=0$, the first two lines have the same y intercept 6. Finding where the third line intersects the first two: $x=-4y-3\Rightarrow y=-8y+6-6\Rightarrow y=0\Rightarrow$ $x=-3$; $y=4y+3+6\Rightarrow y=-3\Rightarrow x=3$; the line $y=-x+6$ intersects the x axis at 6; split the area into two triangles with the base along the x axis: area = $\dfrac{1}{2}\cdot 9\cdot 6+\dfrac{1}{2}\cdot 9\cdot 3=\dfrac{81}{2}$ or 40.5.





3. The line $y=-\dfrac{3}{2}x+3$ intersects the coordinate axes at $A(0,3)$ and $B(2,0)$. The slope of the sides $\perp\overline{AB}$ is $\dfrac{2}{3}$. To locate the other vertices of the square move 3 units to the right and 2 units up $\Rightarrow$ other vertices are $(5,2)$ and $(3,5)$.

ROUND 3 – Geometry: Polygons: Area and Perimeter

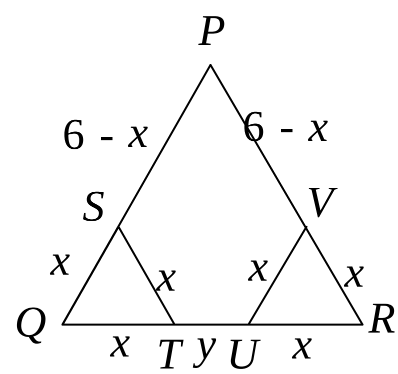


1. Since we are dealing with ratio of areas, let $HG = 1 \Rightarrow GF = 3$. Since $BE = BF$, all the acute angles in the figure measure $45° \Rightarrow DG = \frac{1}{\sqrt{2}}$ and $GC = \frac{3}{\sqrt{2}} \Rightarrow CD = \frac{4}{\sqrt{2}} = 2\sqrt{2} \Rightarrow$ the ratio of the areas = $3 : \left(2\sqrt{2}\right)^2 = 3:8$.

2. $BD = \sqrt{70-54} = 4 \Rightarrow$ area of $\Delta\, ABD$ = $\frac{1}{2}(4)\left(3\sqrt{6}\right) = 6\sqrt{6}$; to find the area of $\Delta\, BCD$, use *Heron's Formula*: $s = 8 \Rightarrow$ area of $\Delta\, BCD = \sqrt{8\cdot 4\cdot 3\cdot 1} = 4\sqrt{6} \Rightarrow$ area of polygon $ABCD$ = $6\sqrt{6} + 4\sqrt{6} = 10\sqrt{6}$.

3. Let $QS = x \Rightarrow PS = 6 - x$; let $TU = y$; the diagram on the right is marked accordingly; the perimeter of $\Delta PQR = 18$ and the perimeter of pentagon $PSTUV$ $= 12 + y \Rightarrow 18 = 12 + y + 5 \Rightarrow y = 1 \Rightarrow 2x + 1 = 6 \Rightarrow x = 2.5$; the ratio of the area of $PSTUV : \Delta PQR$ =

$$\frac{6^2\left(\frac{\sqrt{3}}{4}\right) - 2(2.5)^2\left(\frac{\sqrt{3}}{4}\right)}{6^2\left(\frac{\sqrt{3}}{4}\right)} = \frac{36 - 12.5}{36} = \frac{47}{72}.$$



ROUND 4 – Algebra 2– Logs, Exponents, Radicals and equations involving them

1. $\log(2x+3) - \log(2x+1) = \log x \Rightarrow \log(2x+3) = \log x + \log(2x+1) \Rightarrow$
$\log(2x+3) = \log\left(x(2x+1)\right) \Rightarrow 2x^2 + x = 2x + 3 \Rightarrow 2x^2 - x - 3 = 0 \Rightarrow (2x-3)(x+1) = 0$
$\Rightarrow x = \cancel{-1}, \frac{3}{2} \Rightarrow x = \frac{3}{2}$

2. $\log_b 9 \cdot \log_{27} 100 = 8 \Rightarrow \frac{\log 9}{\log b} \cdot \frac{\log 100}{\log 27} = 8 \Rightarrow \frac{2\log 3}{\log b} \cdot \frac{2}{3\log 3} = 8 \Rightarrow$
$8\log b = \frac{4}{3} \Rightarrow \log b = \frac{1}{6} \Rightarrow b = 10^{1/6} \Rightarrow b^3 = 10^{1/2} = \sqrt{10}$.

3. $6^{2\log_6 x} - \log_3 6^x + \log_3 2^x = 132 \Rightarrow 6^{\log_6 x^2} - \left(\log_3 6^x - \log_3 2^x\right) = 132 \Rightarrow x^2 - \log_3 \frac{6^x}{2^x} = 132$
$x^2 - \log_3 3^x = 132 \Rightarrow x^2 - x - 132 = 0 \Rightarrow (x-12)(x+11) = 0 \Rightarrow x = \cancel{-11}, 12 \Rightarrow x = 12$.

ROUND 5 – Trig. Analysis and Complex Numbers, Trig Form

1. $z\left(1+i\sqrt{3}\right)=-2\sqrt{2}+2i\sqrt{2}$ **Þ** $z\left(2\text{cis}60°\right)=4\text{cis}135° \Rightarrow z=\dfrac{4\text{cis}135°}{2\text{cis}60°}=2\text{cis}75°$

2. $\tan\left(x+\dfrac{p}{3}\right)=4\sqrt{3} \Rightarrow \dfrac{\tan x+\sqrt{3}}{1-\sqrt{3}\tan x}=4\sqrt{3} \Rightarrow \tan x+\sqrt{3}=4\sqrt{3}-12\tan x \Rightarrow$
$13\tan x=3\sqrt{3} \Rightarrow \tan x=\dfrac{3\sqrt{3}}{13}$

3. $\dfrac{z+2}{-2i}=\dfrac{4+4i}{z^2-2z+4} \Rightarrow (z+2)\left(z^2-2z+4\right)=-2i(4+4i) \Rightarrow z^3+8=8-8i \Rightarrow z^3=-8i \Rightarrow$
$z^3=8\ \text{cis}270° \Rightarrow z=2\text{cis}\left(90°+120°n\right)\Big|n=0,1,2 \Rightarrow z=2\text{cis}90°, 2\text{cis}210°, 2\text{cis}330°$.

TEAM ROUND

1. $\dfrac{22}{2x^2+5x-12}+\dfrac{A}{2x^2-9x+9}=\dfrac{22}{(2x-3)(x+4)}+\dfrac{A}{(2x-3)(x-3)}=\dfrac{22(x-3)+A(x+4)}{(2x-3)(x-3)(x+4)}$
For the rational expression to reduce , when $x=1.5$, then $22(x-3)+A(x+4)=0 \Rightarrow$
$-33+5.5A=0 \Rightarrow A=6$.

2. Let the coordinates of point P be $(x,y) \Rightarrow x=\dfrac{(-6)(1)+(6)(3)}{1+3}=\dfrac{12}{4}=3$ and
$y=\dfrac{(3)(1)+(11)(3)}{1+3}=\dfrac{36}{4}=9$. $\overleftrightarrow{AB}$ is $y-3=\dfrac{11-3}{6+6}(x+6) \Rightarrow$ when $y=0$, then
$-3=\dfrac{2}{3}(x+6) \Rightarrow x+6=-\dfrac{9}{2} \Rightarrow x=-\dfrac{21}{2}$; the length of half the base = $3-\left(-\dfrac{21}{2}\right)=\dfrac{27}{2} \Rightarrow$
the length of the base = 27.

3. $\log_x y+\log_y x^9=6+\dfrac{1}{\log\sqrt{y}\cdot\log\sqrt{x}} \Rightarrow \dfrac{\log y}{\log x}+\dfrac{9\log x}{\log y}=6+\dfrac{4}{\log y\cdot\log x} \Rightarrow$
$(\log y)^2+9(\log x)^2=6\log x\cdot\log y+4 \Rightarrow (\log y)^2-6\log x\cdot\log y+9(\log x)^2=4 \Rightarrow$
$(\log y-3\log x)^2=4 \Rightarrow \log\left(\dfrac{y}{x^3}\right)=\pm 2 \Rightarrow \dfrac{y}{x^3}=10^{\pm 2} \Rightarrow y=100x^3$ or $\dfrac{x^3}{100}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2002

ANSWER SHEET:

ROUND 1

1. x

2. $\frac{3}{2}$ $(1.5 \text{ or } 1\frac{1}{2})$

3. 4

ROUND 2

1. – 6

2. $\frac{81}{2}$ $(40.5 \textit{ or } 40\frac{1}{2})$

3. $(5,2),(3,5)$

ROUND 3

1. $3:8$ $\left(\text{or}\,\frac{3}{8}\right)$

2. $10\sqrt{6}$

3. $47:72$ $\left(\text{or } \frac{47}{72}\right)$

ROUND 4

1. $\frac{3}{2}$ $(1.5 \text{ or } 1\frac{1}{2})$

2. $\sqrt{10}$

3. 12

ROUND 5

1. 2 cis 75°

2. $\frac{3\sqrt{3}}{13}$

3. 2 cis 90°, 2 cis 210°,2 cis 330°

TEAM ROUND

3 pts. 1. 6

3 pts. 2. 27

4 pts. 3. $y = 100x^3$ or $0.01x^3$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2003

ROUND 1 – Algebra 1: Fractions and Word Problems

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Simplify the following expression: $\dfrac{x-\dfrac{15}{x-2}}{x+12+\dfrac{45}{x-2}}$, $x \neq 2$.

2. Solve the following equation for x: $\dfrac{1}{6}+\dfrac{4}{3x^3}=\dfrac{1}{2x}+\dfrac{1}{x^2}$, $x \neq 0$.

3. Bob takes twice as long as Joan to build a doll house, while Joe takes 6 hours longer than Joan. Joan can accomplish in 7.5 hours what Bob and Joe can do together in 6 hours. Bob and Joe work 4 hours and stop. Joan finishes building the doll house. How many hours did she work?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2003

ROUND 2 – Coordinate Geometry of the Straight Line

1. (____,____)

2. ___________________

3. ___________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The line $3x+2y=12$ has y-intercept A and x-intercept B. Point P is three-quarters of the distance from A to B. Find the coordinates of point P.

2. A line perpendicular to ℓ whose equation is $3x-4y-8=0$ contains the points $(a,-10)$ and $(-6,a)$. Find the distance from $(-6,a)$ to ℓ.

3. The lines $y=mx+b$, $y=3mx+b$, and $y=4b$, with $m>0$ and $b>0$, form a triangle with an area of A square units. Find m in terms of b and A.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2003

ROUND 3 – Geometry: Polygons: Area and Perimeter

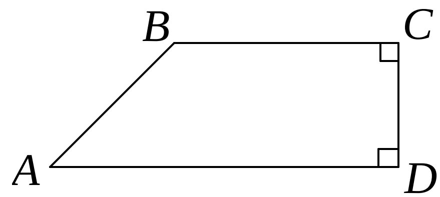
1. ____________________

2. ____________________

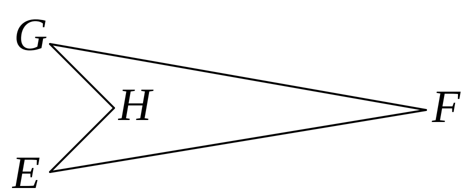
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given $AB = 12$, $m\angle B = 135°$, right angles at vertices C and D, and the area of $ABCD$ = 78, find $BC + CD + AD$.



2. Given $\overline{EH} \perp \overline{HG}$, $EH = HG = 2$, and $EF = FG = 10$, find the area of $EFGH$.



3. $ABCD$ is a rectangle. Equilateral triangles BCE and ADF are drawn such that points E and F are in the interior of the rectangle and the two triangles do not intersect. If $AB = 5\sqrt{3}$ and the area of the non-convex hexagon $ABECDF$ = $9\sqrt{3}$, then the length of $\overline{BC}$ can be written in the form $a - \sqrt{b}$, where a and b are positive integers. Find the product of a and b.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2003

ROUND 4 – Algebra 2– Logs, Exponents, Radicals and equations involving them

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given $\log_b 9 = M$ and $\log_b 8 = N$, find $\log_b\left(\frac{27}{16}\right)$ in terms of M and N

2. Find the value of $\log_{18}\sqrt{2} - \log_{18}\sqrt[3]{2} + \log_{18}\sqrt[3]{3}$.

3. Given $\log_a \sqrt[3]{x} - \log_{a^2}\sqrt{x} = \log_x a^3$, $a > 0$, $a \neq 1$, solve for x in terms of a.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2003

ROUND 5 – Trig. Analysis and Complex Numbers, Trig Form

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

Note that $r\,cis\,q = r\left(\cos q + i\sin q\right)$ where $i = \sqrt{-1}$.

1. Given $\tan q = -\dfrac{\sqrt{7}}{\sqrt{2}}$ and $\cos q > 0$, find the numerical value of $1 + \cos 2q$.

2. Given z is in the 2nd quadrant of the complex plane, $z^3 = \left(-4\sqrt{3} + 4i\right)^2$ and $w = \dfrac{z}{1+i}$, find w in $r\,\mathrm{cis}\,q$ form where $r > 0$ and $0° \le q < 360°$.

3. Given ΔABC with $AB = 4$, $BC = 5$, $AC = 6$, $\overline{DC} \perp \overline{BC}$, and point D is on the bisector of $\angle B$, find the length of $\overline{BD}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2003

TEAM ROUND: Time Limit: 12 Minutes

3 pts. 1. ___________________

3 pts. 2. ___________________

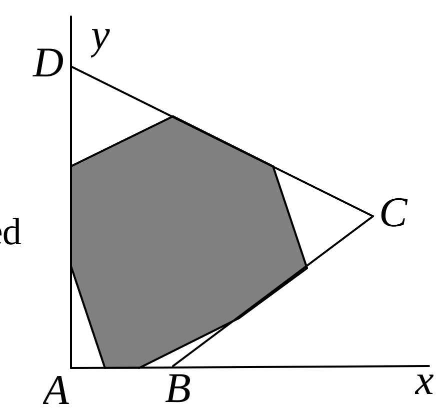
4 pts. 3. ___________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given $\log_b 24 = x$ and $\log_b 54 = y$, find $\log_b 288$ in terms of *x* and *y*.

2. A right triangle has an area of 24 and the length of the bisector of the right angle is 4. Find the length of its hypotenuse.

3. Quadrilateral *ABCD* on the right is constructed by the intersections of the lines $3x - 4y = 6,\ x + 2y = 12$, and the coordinate axes. Each of its sides is trisected and the trisection points are connected forming the shaded octagon Find the area of this octagon.



Detailed solutions to GBML Meet 3 2003

ROUND 1 – Algebra 1: Fractions and Word Problems

1. $$\frac{x-\frac{15}{x-2}}{x+12+\frac{45}{x-2}}=\frac{x^2-2x-15}{x^2+10x+21}=\frac{(x-5)(x+3)}{(x+7)(x+3)}=\frac{x-5}{x+7}$$

2. $$\frac{1}{6}+\frac{4}{3x^3}=\frac{1}{2x}+\frac{1}{x^2}\Rightarrow 6x^3\left(\frac{1}{6}+\frac{4}{3x^3}\right)=6x^3\left(\frac{1}{2x}+\frac{1}{x^2}\right)\Rightarrow x^3+8=3x^2+6x\Rightarrow$$
$(x+2)(x^2-2x+4)=3x(x+2)\Rightarrow x=-2$ or $x^2-2x+4=3x\Rightarrow$
$x^2-5x+4=0\Rightarrow(x-1)(x-4)=0\Rightarrow$ solutions for x are $-2,1,4$

3. Let x = hours for Joan to build the doll house $\Rightarrow 2x$ = hours for Bob and $x+6=$ hours for Joe $\Rightarrow \frac{7.5}{x}=\frac{6}{2x}+\frac{6}{x+6}\Rightarrow\frac{15}{2x}=\frac{6}{2x}+\frac{6}{x+6}\Rightarrow\frac{9}{2x}=\frac{6}{x+6}\Rightarrow 9x+54=12x\Rightarrow$
$3x=54\Rightarrow x=18$; in 4 hours Bob and Joe build $\frac{4}{36}+\frac{4}{24}=\frac{5}{18}$ of the house $\Rightarrow$ there is $\frac{13}{18}$ left to build which will take Joan 13 hours.

ROUND 2 – Coordinate Geometry of the Straight Line

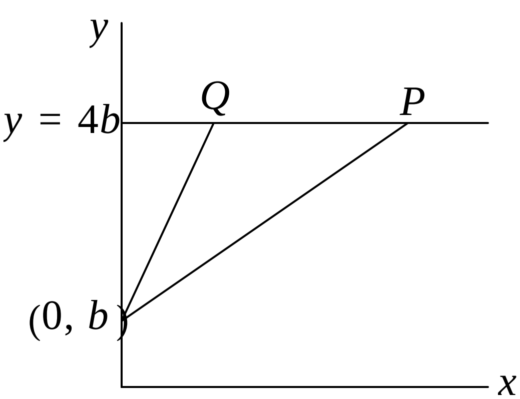
1. $3x+2y=12$ has y-intercept $A(0,6)$ and x-intercept $B(4,0)$. Point P, three-quarters of the way from A to B, has coordinates $(3,1.5)$.

2. The slope of $3x-4y-8=0$ is $\frac{3}{4}\Rightarrow$ slope of the perpendicular = $-\frac{4}{3}\Rightarrow$
$\frac{a+10}{-6-a}=-\frac{4}{3}\Rightarrow 3a+30=4a+24\Rightarrow a=6$. The distance from $(-6,6)$ =
$$\frac{|3(-6)-4(6)-8|}{\sqrt{3^2+(-4)^2}}=\frac{50}{5}=10.$$

3. Point P's x-coordinate: $4b=mx+b\Rightarrow x=\frac{3b}{m}$;

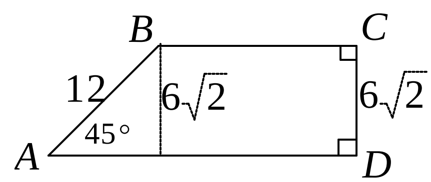
Point Q's x-coordinate: $4b=3mx+b\Rightarrow x=\frac{b}{m}$;

$$PQ=\frac{2b}{m}\Rightarrow\frac{1}{2}\left(\frac{2b}{m}\right)(3b)=A\Rightarrow m=\frac{3b^2}{A}.$$

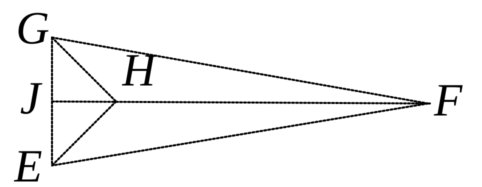


ROUND 3 – Geometry: Polygons: Area and Perimeter

1. The height of the trapezoid = $6\sqrt{2} \Rightarrow$
$3\sqrt{2}(BC + AD) = 78 \Rightarrow BC + AD = 13\sqrt{2} \Rightarrow$
$BC + CD + AD = 19\sqrt{2}$



2. Draw $\overline{GE}$ and altitude $\overline{FJ}$ to ΔGFE.
$GE = 2\sqrt{2} \Rightarrow GJ = JH = \sqrt{2} \Rightarrow FJ = \sqrt{100-2} =$
$\sqrt{98} = 7\sqrt{2} \Rightarrow FH = 6\sqrt{2} \Rightarrow$ area of $EFGH =$
$2\Delta FGH = \left(6\sqrt{2}\right)\left(\sqrt{2}\right) = 12$.



3. The area of the hexagon = area of rectangle – 2×area of the equilateral triangle;
let $x = BC \Rightarrow 5\sqrt{3}x - \dfrac{x^2\sqrt{3}}{2} = 9\sqrt{3} \Rightarrow x^2 - 10x + 18 = 0 \Rightarrow (x-5)^2 = 7 \Rightarrow$
$x - 5 = -\sqrt{7} \Rightarrow x = 5 - \sqrt{7} \Rightarrow ab = 35$. Note $x = 5 + \sqrt{7}$ is not possible because $\left(5+\sqrt{7}\right)\sqrt{3} > 5\sqrt{3}$ and the equilateral triangles would intersect as a result.

ROUND 4 – Algebra 2– Logs, Exponents, Radicals and equations involving them

1. $\log_b 9 = M \Rightarrow \log_b 3 = \dfrac{M}{2}, \log_b 8 = N \Rightarrow \log_b 2 = \dfrac{N}{3}$;
$\log_b\left(\dfrac{27}{16}\right) = 3\log_b 3 - 4\log_b 2 = \dfrac{3M}{2} - \dfrac{4N}{3} = \dfrac{9M - 8N}{6}$.

2. $\log_{18}\sqrt{2} - \log_{18}\sqrt[3]{2} + \log_{18}\sqrt[3]{3} = \dfrac{1}{2}\log_{18} 2 - \dfrac{1}{3}\log_{18} 2 + \dfrac{1}{3}\log_{18} 3 =$
$\dfrac{1}{6}\log_{18} 2 + \dfrac{1}{6}\log_{18} 3^2 = \dfrac{1}{6}\log_{18}\left(2 \cdot 3^2\right) = \dfrac{1}{6}\log_{18} 18 = \dfrac{1}{6}$

3. $\log_a \sqrt[3]{x} - \log_{a^2}\sqrt{x} = \log_x a^3 \Rightarrow \dfrac{\log x}{3\log a} - \dfrac{\log x}{4\log a} = \dfrac{3\log a}{\log x} \Rightarrow \dfrac{\log x}{12\log a} = \dfrac{3\log a}{\log x} \Rightarrow$
$(\log x)^2 = 36(\log a)^2 \Rightarrow \log x = \pm 6\log a \Rightarrow x = a^{-6}, a^6$

ROUND 5 – Trig. Analysis and Complex Numbers, Trig Form

1. $1+\cos 2q = 1+2\cos^2 q - 1 = 2\cos^2 q = \dfrac{2}{\sec^2 q} = \dfrac{2}{1+\tan^2 q} = \dfrac{2}{1+7/2} = \dfrac{4}{9}$

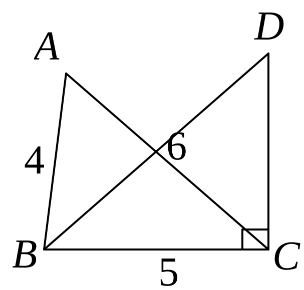
2. $z^3 = \left(-4\sqrt{3}+4i\right)^2 \Rightarrow z^3 = \left(8cis150°\right)^2 \Rightarrow z^3 = 64cis300° \Rightarrow z = 64^{1/3}cis\left(100°+120°k\right)$

where $k=0,1,2$; since z is in quadrant II $\Rightarrow$ $z = 64^{1/3}cis100° = 4\,cis100°$;

$w = \dfrac{4cis100°}{1+i} = \dfrac{4cis100°}{\sqrt{2}cis45°} = 2\sqrt{2}\,cis55°$

3. Let $\angle ABC = q$: $\cos q = \dfrac{4^2+5^2-6^2}{2\cdot 4\cdot 5} = \dfrac{1}{8} \Rightarrow$

$\cos\left(\dfrac{q}{2}\right) = \sqrt{\dfrac{1+1/8}{2}} = \dfrac{3}{4}$; $\dfrac{5}{BD} = \dfrac{3}{4} \Rightarrow BD = \dfrac{20}{3}$.



TEAM ROUND

1. $\log_b 24 = x \Rightarrow 3\log_b 2 + \log_b 3 = x$ and $\log_b 54 = 9 \Rightarrow \log_b 2 + 3\log_b 3 = y \Rightarrow$

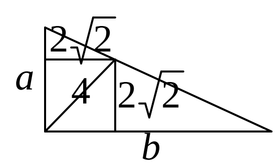
$\log_b 2 = \dfrac{3x-y}{8}$ and $\log_b 3 = \dfrac{3y-x}{8}$; $\log_b 288 = 5\log_b 2 + 2\log_b 3$

$= 5\left(\dfrac{3x-y}{8}\right) + 2\left(\dfrac{3y-x}{8}\right) = \dfrac{13x+y}{8}$.

2. Let the length of the legs = a and b; since the area = 24 $\Rightarrow$

$\dfrac{ab}{2} = 24$ and $\dfrac{2\sqrt{2}a}{2} + \dfrac{2\sqrt{2}b}{2} = 24 \Rightarrow ab = 48$ and $a+b = 12\sqrt{2}$

$a^2+b^2 = (a+b)^2 - 2ab = \left(12\sqrt{2}\right)^2 - 2(48) = 192 \Rightarrow$ hypotenuse = $\sqrt{192} = 8\sqrt{3}$

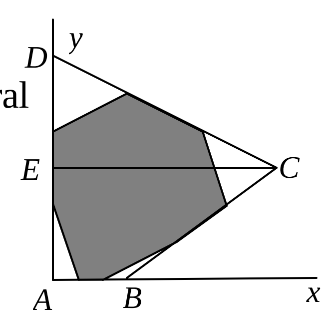


3. $B(2,0)$, $D(0,6)$, $C(6,3)$; the area of $ABCD$ =

$\dfrac{1}{2}(3)(2+6) + \dfrac{1}{2}(6)(3) = 21$**;** the triangles cut off the quadrilateral

at B and D are $\dfrac{1}{9}$ the area of $ABCD$; likewise for the triangles

at A and C; therefore the area of the octagon = $\dfrac{7}{9}(21) = \dfrac{49}{3}$;



GREATER BOSTON MATHEMATICS LEAGUE
MEET 3 – DECEMBER 2003

ANSWER KEY:

ROUND 1

1. $\dfrac{x-5}{x+7}$

2. $-2,1,4$

3. 13

ROUND 2

1. $\left(3, \dfrac{3}{2}\right)$ or equivalent

2. 10

3. $\dfrac{3b^2}{A}$

ROUND 3

1. $19\sqrt{2}$

2. 12

3. 35

ROUND 4

1. $\dfrac{9M-8N}{6}$ $\left(\text{or } \dfrac{3}{2}M - \dfrac{4}{3}N\right)$

2. $\dfrac{1}{6}$

3. a^{-6}, a^{6} (or $a^{\pm 6}$ or $a^{6}, \dfrac{1}{a^{6}}$)

ROUND 5

1. $\dfrac{4}{9}$

2. $2\sqrt{2}\, cis\, 55°$

3. $\dfrac{20}{3}$ (or $6\frac{2}{3}$)

TEAM ROUND

3 pts. 1. $\dfrac{13x+y}{8}$ $\left(\text{or } \dfrac{13}{8}x + \dfrac{1}{8}y\right)$

3 pts. 2. $8\sqrt{3}$

4 pts. 3. $\dfrac{49}{3}$ $\left(\text{or } 16\frac{1}{3}\right)$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 1999

ROUND 1 – Volume and Surface Area of Solids

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The total surface area of a cube is 72 square inches. Find the number of inches in the length of one diagonal of the cube. Write the result in simplest radical form.

2. A regular triangular prism has a volume of $12\sqrt{3}$ cm^3. If the height of the prism equals the perimeter of the base, find the number of centimeters in the height of the prism. Write the result in simplest radical form.

3. A sphere with a radius of length six is inscribed in a right circular cone with a height of length fifteen. Find the volume interior to the cone and exterior to the sphere.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 1999

ROUND 2 – Inequalities and Absolute Value

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Solve the following equation for x: $6x^2 = 19|x| - 10$

2. **How many integers** satisfy the following system of inequalities?

$|3x + 8| < 23$ and $|4x - 2| \geq 10$

3. Solve the following inequality for x: $\dfrac{3}{2x - 8} \geq \dfrac{x + 2}{x^2 - 4x}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 1999

ROUND 3 – Similar Polygons, Circles and Areas Related to Circles

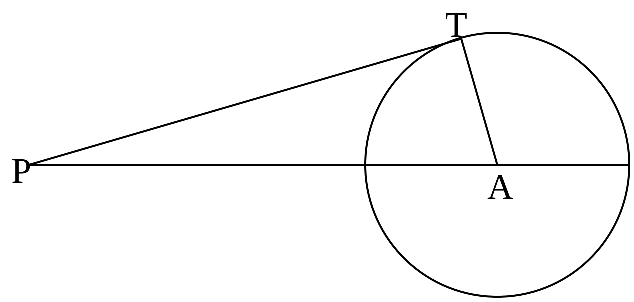
1. ____________________

2. ____________________

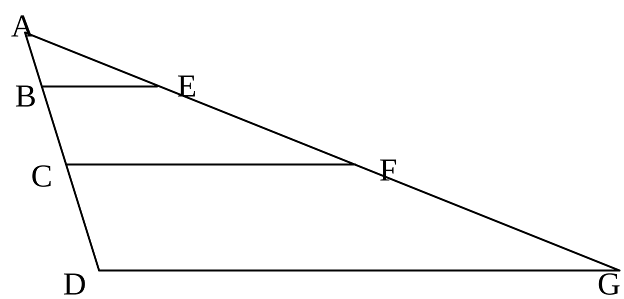
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. From point P, 9 inches from the closest point on a circle centered at A with diameter of length 7 inches, one tangent is drawn to the circle with T, the point of tangency. Find the number of square inches in the area of Δ PAT.



2. Given $\overline{BE}$ // $\overline{CF}$ // $\overline{DG}$, AB:BC:CD = 2:3:4, and the area of BEFC = 126, find the area of CFGD.



3. Given Δ RST with $\overline{RS} \perp \overline{ST}$, RS = 6, and ST = 8, find the area of the region interior to the circumscribed circle and exterior to the inscribed circle of Δ RST.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 1999

ROUND 4 – Sequences and Complex Numbers

1. ____________________

2. ____________________

3. (____ , ____ , ____)

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the arithmetic sequence, $5 + 4i$, $7 + i$, $9 - 2i$, …, find the sum of its first 20 terms.

2. Given the geometric sequence where $a_1 = 2$ and $r = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, find a_{1999}.

3. Given the following sequence of **positive numbers**, 4, x, y, z, 100, where the first three numbers form a geometric sequence, the middle three numbers form an arithmetic sequence, and the last three numbers form a geometric sequence, find the ordered triple, (x, y, z).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 1999

ROUND 5 – Conics

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find in simplest radical form the distance from the center of $x^2 + y^2 + 6x - 16y - 3 = 0$ to the vertex of $y^2 + 4y - 5x + 14 = 0$

2. Given the ellipse $4x^2 + 9y^2 = 36$, find the focus of the parabola whose vertex is at the lower y-intercept of the ellipse and which passes through the x-intercepts of the ellipse.

3. Given the conic with foci (7, -3) and (-1, -3) such that the difference of the distances from any point on this conic to the foci is 4, find the distance between its x intercepts in simplest radical form.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 1999

TEAM ROUND
Problem submitted by Maimonides

3 pts. 1. ___________________

3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. Four letters are chosen randomly from the word *MATHEMATICS*. What is the probability the letters chosen can be used to spell the word *MATH* ? [For example, *TAHM* can be used to spell *MATH*.] Write the answer in the form $\frac{a}{b}$ where a and b are relatively prime whole numbers.

2. Given the five positive numbers, 17, 4, 28, 23, and x, such that their mean equals their median, find all possible values for x.

3. From a standard deck of playing cards (no jokers), two cards are chosen at random and from a box containing four red, three blue and two white marbles, two marbles are chosen at random. What is the probability that at least one of the cards is a face card and the two marbles chosen are of different colors? Write the answer in the form $\frac{a}{b}$ where a and b are relatively prime whole numbers.

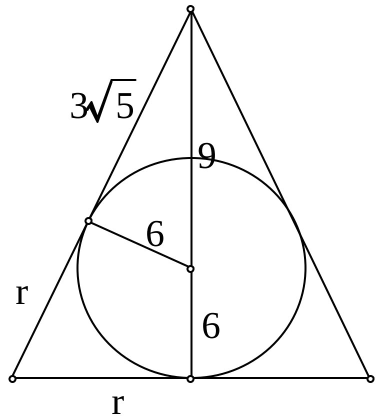
Detailed Solutions of GBML for MEET 4 – JANUARY 1999

ROUND 1

1. If x is the length of an edge of the cube, the total surface area = $6\,x^2 = 72 \Rightarrow$
$x = 2\sqrt{3}$. The length of the diagonal = $x\sqrt{3} = 2\sqrt{3}\cdot\sqrt{3} = \mathbf{6}\ in.$

2. The base of the prism is an equilateral triangle. Call the length of one side x. $\Rightarrow$
$A = \dfrac{x^2\sqrt{3}}{4}$ and $h = 3x \Rightarrow V = \dfrac{x^2\sqrt{3}}{4}\cdot 3x = 12\sqrt{3} \Rightarrow x^3 = 16 \Rightarrow x = 2\sqrt[3]{2} \Rightarrow h = \mathbf{6\sqrt[3]{2}}$

3. $r^2 + 15^2 = \left(r + 3\sqrt{5}\right)^2 \Rightarrow r^2 + 225 = r^2 + 6\sqrt{5}\ r + 45 \Rightarrow$
$r = 6\sqrt{5}$; $V = \frac{1}{3}\left(6\sqrt{5}\right)^2 15\,\pi - \frac{4}{3}\cdot 6^3\,\pi = \mathbf{612p}$



ROUND 2

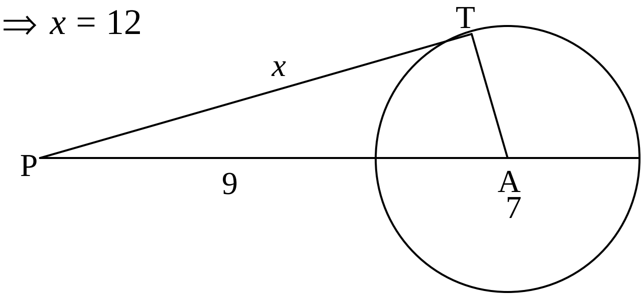
1. $6x^2 = 19\left|x\right| - 10 \Rightarrow 6\left|x\right|^2 - 19\left|x\right| + 10 = 0 \Rightarrow \left(3\left|x\right| - 2\right)\left(2\left|x\right| - 5\right) = 0 \Rightarrow$ the solutions
for x are $\pm\,\mathbf{\dfrac{2}{3}}$ and $\pm\,\mathbf{\dfrac{5}{2}}$

2. $\left|3x + 8\right| < 23$ and $\left|4x - 2\right| \geq 10 \Rightarrow -23 < 3x + 8 < 23$ and $4x - 2 \geq 10$ or $4x - 2 \leq -10$
$\Rightarrow -10\frac{1}{3} < x < 5$ and $x \geq 3$ or $x \leq -2 \Rightarrow x = -10, -9, \ldots -2, 3, 4$ which are **11** possibilities.

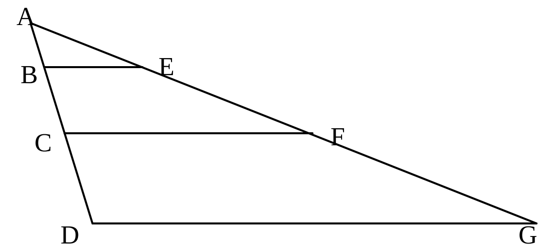
3. $\dfrac{3}{2x - 8} \geq \dfrac{x + 2}{x^2 - 4x} \Rightarrow \dfrac{3}{2(x - 4)} - \dfrac{x + 2}{x(x - 4)} \geq 0 \Rightarrow \dfrac{3x - 2(x + 2)}{2x(x - 4)} \geq 0 \Rightarrow \dfrac{1}{2x} \geq 0$ and $x \neq 4$
$\Rightarrow$ ***x* > 0 and *x* ¹ 4**

ROUND 3

1. To find the tangent segment: $x^2 = 9 \cdot 16 \Rightarrow x = 12$
$\Rightarrow$ area of Δ PAT $= \left(\frac{1}{2}\right)\left(\frac{7}{2}\right)(12) = \mathbf{21}\ in^2$



2. AB:BC:CD = 2:3:4 $\Rightarrow$ AB:AC:AD = 2:5:9 $\Rightarrow$
Δ ABE: ΔACF: ΔADG = 4:25:81 $\Rightarrow$
Δ ABE = $4x$, Δ ACF = $25x$, and Δ ADG = $81x \Rightarrow$
BEFC = 126 = $21x \Rightarrow x = 6$ and CFGD = $56x$ = **336**



3. The radius of the circumscribed circle = half the hypotenuse = 5. To find the radius of the inscribed circle, use the formula $\frac{1}{2}Pr = A \Rightarrow 12r = 24$, and $r = 2 \Rightarrow$ area of the region = $25\pi - 4\pi$ = $\mathbf{21\pi}$

ROUND 4

1. For the arithmetic sequence, $5 + 4i,\ 7 + i,\ 9 - 2i,\ \ldots, \Rightarrow d = 2 - 3i$. Now use the formula for arithmetic series, $S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right) \Rightarrow S_{20} = 10\left(2(5+4i) + 19(2-3i)\right) = \mathbf{480 - 490\mathit{i}}$

2. $a_1 = 2$ and $r = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $r^2 = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = 2\left(\frac{\sqrt{2}}{2}\right)^2 i = i \Rightarrow r^4 = -1 \Rightarrow r^8 = 1$;
$a_{1999} = a_1 \cdot r^{1998}$ and 1998 = 6 mod 8 $\Rightarrow a_{1999} = 2 \cdot r^6 = 2 \cdot r^4 \cdot r^2 = 2(-1)i =$ $\mathbf{-2\mathit{i}}$

3. Since 4, x, y form a geom. seq. $\Rightarrow x^2 = 4y$; since x, y, z form an arith. seq. $\Rightarrow x + z = 2y$; since y, z, 100 form a geom. seq. $\Rightarrow z^2 = 100y$; $x^2 = 4y \Rightarrow$
$25x^2 = 100y \Rightarrow z^2 = 25x^2 \Rightarrow z = 5x$ (both positive); $x + z = 2y \Rightarrow$
$x + 5x = \frac{x^2}{2} \Rightarrow x^2 = 12x \Rightarrow x = 12\ (x \neq 0) \Rightarrow z = 60 \Rightarrow y = 36 \Rightarrow$ **(12, 36, 60)** is the answer

ROUND 5

1. $x^2 + y^2 + 6x - 16y - 3 = 0 \Rightarrow x^2 + 6x + 9 + y^2 - 16y + 64 = 76 \Rightarrow (x + 3)^2 + (y - 8)^2 = 76$ $\Rightarrow$ center = (-3, 8)
$y^2 + 4y - 5x + 14 = 0 \Rightarrow y^2 + 4y + 4 = 5x - 14 + 4 \Rightarrow (y + 2)^2 = 5(x - 2) \Rightarrow$
vertex = (2, -2) . Distance between these points = $\sqrt{5^2 + (-10)^2} = \mathbf{5\sqrt{5}}$

2. The vertex of the parabola is (0, -2) and the x-intercepts are (3, 0) and (-3, 0) $\Rightarrow$
Equation of the parabola is $x^2 = 4p(y + 2) \Rightarrow$ Since $(3, 0)$ is a point on the parabola, then
$9 = 8p$ and $p = \frac{9}{8} \Rightarrow$ $focus = \left(0, -2 + \frac{9}{8}\right) = \left(\mathbf{0, -\frac{7}{8}}\right)$

3. foci (7, -3) and (-1, -3) and the diff. of dist. = 4 $\Rightarrow$ hyperbola has center (3, -3) and
$2a = 4 \Rightarrow a = 2$; $2c = 8 \Rightarrow c = 4 \Rightarrow b^2 = c^2 - a^2 = 12 \Rightarrow$ equation of the hyperbola is
$\frac{(x-3)^2}{4} - \frac{(y+3)^2}{12} = 1$; $y = 0$: $\frac{(x-3)^2}{4} - \frac{9}{12} = 1 \Rightarrow (x - 3)^2 = 7 \Rightarrow x = 3 \pm \sqrt{7} \Rightarrow$
distance between the x intercepts = $\mathbf{2\sqrt{7}}$

TEAM ROUND

1. The probability the letters will be chosen in any order $MATH = \frac{2 \cdot 2 \cdot 2 \cdot 1}{\binom{11}{4}} = \mathbf{\frac{4}{165}}$

2. The mean of 4, 17, 23, 28, and x is $\frac{72 + x}{5}$; if $x \le 17$, then $\frac{72 + x}{5} = 17 \Rightarrow x = 13$;
If $17 < x \le 23$, then $\frac{72 + x}{5} = x \Rightarrow x = 18$; If $x > 23$, then $\frac{72 + x}{5} = 23 \Rightarrow x = 43. \Rightarrow$
Answer is x = **13, 18, or 43**

3. Probability of choosing at least 1 face card = $\frac{\binom{12}{1}\binom{40}{1} + \binom{12}{2}}{\binom{52}{2}}$

Probability of choosing 2 different colored marbles = $\frac{\binom{4}{1}\binom{3}{1} + \binom{4}{1}\binom{2}{1} + \binom{3}{1}\binom{2}{1}}{\binom{9}{2}}$

$$\frac{\binom{12}{1}\binom{40}{1} + \binom{12}{2}}{\binom{52}{2}} \times \frac{\binom{4}{1}\binom{3}{1} + \binom{4}{1}\binom{2}{1} + \binom{3}{1}\binom{2}{1}}{\binom{9}{2}} = \mathbf{\frac{91}{306}}$$

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 1999

ANSWER SHEET:

ROUND 1

1. 6
2. $6\sqrt[3]{2}$
3. 612π

ROUND 2

1. $\pm\frac{2}{3}$ and $\pm\frac{5}{2}$
2. 11
3. $x > 0$ and $x \neq 4$

ROUND 3

1. 21
2. 336
3. 21π

ROUND 4

1. $480 - 490i$
2. $-2i$ [$0 - 2i$ or $0 + (-2i)$ are acceptable]
3. (12, 36, 60)

ROUND 5

1. $5\sqrt{5}$
2. $\left(0, -\frac{7}{8}\right)$
3. $2\sqrt{7}$

TEAM ROUND

3 pts. 1. $\frac{4}{165}$

3 pts. 2. 13, 18, 43

4 pts. 3. $\frac{91}{306}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2000

ROUND 1 – Volume and Surface Area of Solids

1. ____________________

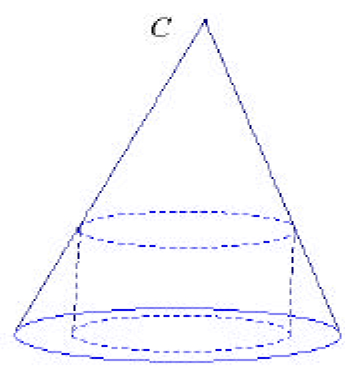
2. ____________________

3. ____________________

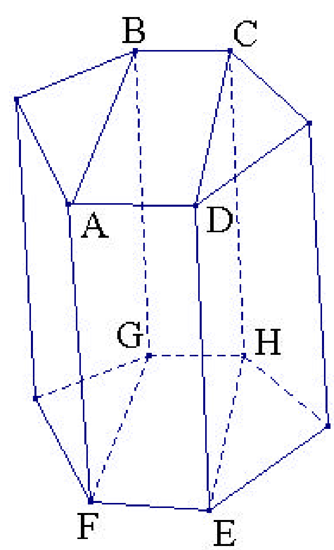
CALCULATORS ARE NOT ALLOWED ON THIS ROUND
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE

1. A rectangular parallelepiped with dimensions 7cm. by 5cm. by 1 cm. has a diagonal the same length as the diagonal of a cube. Find the volume of the cube in cubic centimeters.

2. A right circular cylinder with a radius of 4 inches and a height of 3 inches is inscribed in right circular cone *C* with a radius of 6 inches. (See the figure.) Find the volume in cubic inches of cone *C*.



3. A regular hexagonal right prism has a height of $6\sqrt{3}$ cm. (See the figure.) If the volume of <u>rectangular prism</u> ABCDEFGH is 162 cm^3, find the number of square centimeters in the total surface area of the hexagonal prism.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2000

ROUND 2 – Inequalities and Absolute Value

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find **how many integers** solve the following system of inequalities:

$$\{x \mid |x| < 13 \text{ and } 1 - 3x > 8\}$$

2. Solve the following inequality for *x*:

$$\left\{x \mid \frac{2}{3x} \leq \frac{1}{x-1}\right\}$$

3. Solve the following equation for *x*:

$$\left\{x \mid \left|\sqrt{4x^2 - 12x + 9} - 18\right| = 7\right\}$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2000

ROUND 3 – Similar Polygons, Circles and Areas Related to Circles

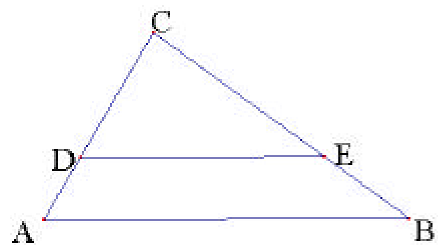
1. ____________________

2. ____________________

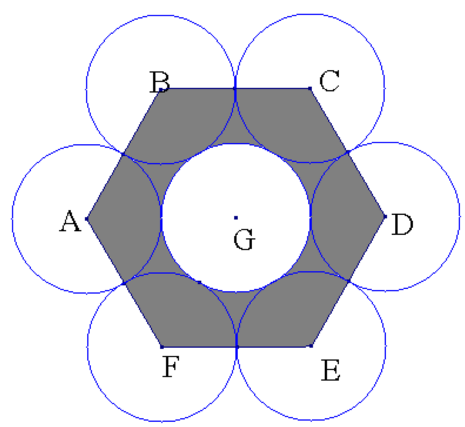
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

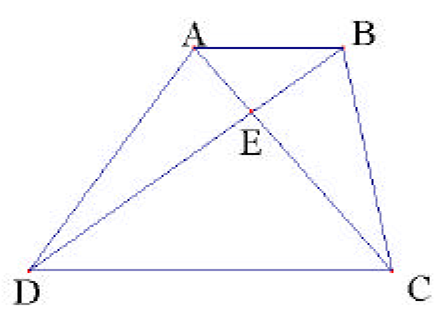
1. Given $\overline{DE}$ parallel to $\overline{AB}$, CD:DA = 2:1, and the area of trapezoid ABED = 45 cm^2, find the number of square centimeters in the area of Δ CDE.



2. Given congruent circles of radius 2 cm. tangent externally in pairs, whose centers form the regular hexagon, ABCDEF, and circle G tangent to all six circles, find the shaded area. (See the figure.)



3. Given $\overline{AB}$ parallel to $\overline{CD}$, and AE:EC = 2:5, find the ratio of the area of Δ ABE to the area of trapezoid ABCD.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2000

ROUND 4 – Sequences and Complex Numbers

1. ____________________

2. ____________________

3. _____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given a 100 term arithmetic series whose sum is 1800, and whose last term is 3 times its first term, find its last term.

2. Given a geometric sequence whose third term is $-4 + 8i$ and whose fourth term is $-16 - 8i$, find its first term.

3. Find the following sum of complex numbers:

$$\sum_{k=1}^{22}\left(5i^{k} + 2i^{3k}\right)$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2000

ROUND 5 – Conics

1. __(,)__

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the coordinates of the focus of the parabola, $\{(x, y) \mid y^2 - 16y + 6x + 79 = 0\}$

2. Given the circle, $\{(x, y) \mid x^2 + y^2 - 12x + 12y + 36 = 0\}$, a tangent line is drawn from the point $P\ (15, -6)$ to this circle. Find the distance from P to the point of tangency.

3. Given the ellipse, $\left\{(x, y) \mid \frac{(x+2)^2}{16} + \frac{(y-4)^2}{25} = 1\right\}$, a hyperbola's foci are the ellipse's vertices and the hyperbola's vertices are the ellipse's foci. Find the y-coordinates of the points on the hyperbola whose x-coordinate is 6.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2000

TEAM ROUND

3 pts. 1. ___________________

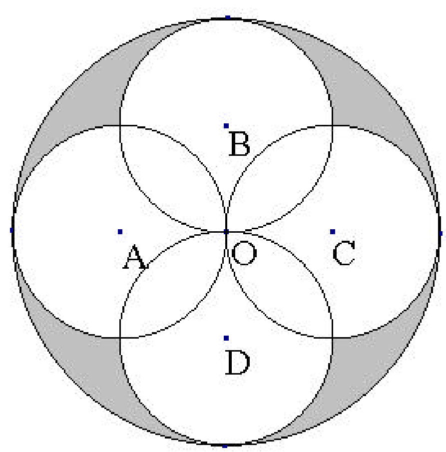
3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. What is the probability if three dice are shaken well and thrown, that the sum of the pips (numbers) on the three top faces either are less than five or greater than fourteen? Express the probability in the rational form, $\frac{a}{b}$, where a and b are relatively prime whole numbers or if estimated, rounded to **4 decimal places**.

2. A circle, centered at O, has a radius of 4 cm. and congruent circles, centered at A, B, C, and D, all contain point O and are tangent internally to circle O. Points A, B, C, and D form a square. (See the figure.) Find the exact shaded area of the figure or if estimated, then rounded to **four decimal places**.



3. Five cards are chosen at random from a standard deck of playing cards containing no jokers. What is the probability that at least 3 out of 5 are of the same suit? Write the answer in decimal form rounded to **4 decimal places**.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2000

ANSWER SHEET:

ROUND 1

1. 125
2. 108π
3. $135\sqrt{3}$

ROUND 2

1. 10
2. $-2 \le x < 0$ or $x > 1$
3. –11, – 4, 7, 14 (in any order)

ROUND 3

1. 36
2. $24\sqrt{3} - 4p \quad \left(4\left(6\sqrt{3} - p\right)\right)$
3. 4:49 $\left(\frac{4}{49}\right)$

ROUND 4

1. 27
2. $1 - 2i$
3. $-7 + 3i$

ROUND 5

1. $(-4, 8)$
2. $3\sqrt{5}$
3. $4 \pm 3\sqrt{5}$

TEAM ROUND

3 pts. 1. $\frac{1}{9}$ (0.1111)

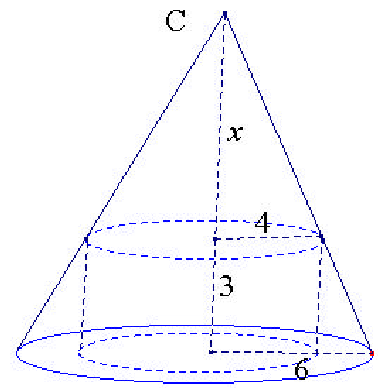
3 pts. 2. $8\pi - 16$ ($8(\pi - 2)$ or 9.1327)

4 pts. 3. 0.3711

Detailed Solutions to GBML Meet 4, January 2000

Round 1

1. diagonal of rectangular parallelepiped = $\sqrt{7^2+5^2+1^2}=\sqrt{75}=5\sqrt{3}\Rightarrow$
edge of the cube = 5 cm. $\Rightarrow$ volume of the cube = 125cm.^{3}



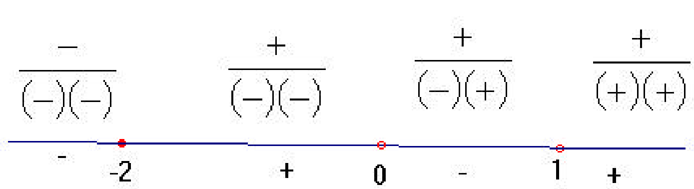
2. $\dfrac{x}{4}=\dfrac{x+3}{6}\Rightarrow x=6\Rightarrow V=\dfrac{1}{3}p\cdot 6^2\cdot 9=108p\ in.^3$

3. AB:AD = $\sqrt{3}$:1 $\Rightarrow$ If AD = x, volume of ABCDEFGH =
$(x)\left(x\sqrt{3}\right)\left(6\sqrt{3}\right)=162\Rightarrow x^2=9\Rightarrow x=3\text{cm.}\Rightarrow$
total surface area of the hexagonal prism =
2×area of hexagon + perimeter of hexagon ×height of prism
= $2\cdot\dfrac{6\cdot 3^2\sqrt{3}}{4}+18\cdot 6\sqrt{3}=27\sqrt{3}+108\sqrt{3}=135\sqrt{3}$ cm.^{2}

Round 2

1. $-13<x<13\Rightarrow x=-12,\ -11,\ldots,11,\ 12;\ 1-3x>8\Rightarrow -3x>7\Rightarrow x<-\dfrac{7}{3}\Rightarrow x=-3,-4,\ldots$
The intersection of these two sets of integers = – 12, – 11, …– 3, which has 10 solutions.

2. $\dfrac{2}{3x}-\dfrac{1}{x-1}\le 0\Rightarrow\dfrac{2x-2-3x}{3x(x-1)}\le 0\Rightarrow\dfrac{-x-2}{3x(x-1)}\le 0\Rightarrow\dfrac{x+2}{3x(x-1)}\ge 0\Rightarrow$ key numbers are:
-2 (included) 0 and 1 (excluded); on the number line:



$\Rightarrow$ solution is $\{x\,|\,-2\le x<0 \text{ or } x>1\}$

3. $\left|\sqrt{4x^2-12x+9}-18\right|=7\Rightarrow\left|\sqrt{(2x-3)^2}-18\right|=7\Rightarrow\left||2x-3|-18\right|=7\Rightarrow$
$|2x-3|-18=\pm 7\Rightarrow|2x-3|=11\ or\ 25\Rightarrow 2x-3=\pm 11\ or\ \pm 25\Rightarrow$
$x=\dfrac{3\pm 11}{2}\ or\ \dfrac{3\pm 25}{2}\Rightarrow x=-11,\ -4,7,14$

Round 3

1. Since CD:DA = 2:1 ⇒ CD:CA= 2:3 ⇒ area of Δ CDE: area of Δ CAB = 4:9; let area of Δ CDE = 4*x*, then area of Δ CAB = 9*x* ⇒ area of ABED = 5*x* = 45 ⇒ *x* = 9 and area of Δ CDE = 36 cm.^{2}

2. The length of long diagonal of the regular hexagon = twice the length of its side ⇒ AD = 8 cm. ⇒ radius of circle G = 2 cm. Shaded area = area of ABCDEF – area of circle G = $6\frac{4^2\sqrt{3}}{4} - p \cdot 2^2 = 24\sqrt{3} - 4p$ cm.^{2}

3. Since AE:EC = 2:5 ⇒ area of Δ AEB: area of Δ CED = 4:25 and area of Δ AEB: area of Δ BEC = 2:5; BE:ED = 2:5 ⇒ area of Δ AEB: area of Δ AED = 2:5; let area of Δ AEB = 4*x* ⇒ area of Δ DEC = 25*x* , area of Δ AED = 10*x*, and area of Δ BEC = 10*x*; area of trapezoid ABCD = $4x + 25x + 10x + 10x = 49x$ ⇒ area of Δ AEB: trapezoid ABCD = 4:49

Round 4

1. $S_n = \frac{n}{2}(a_1 + a_n) \Rightarrow 1800 = \frac{100}{2}(x + 3x) \Rightarrow 200x = 1800 \Rightarrow x = 9 \Rightarrow 3x = 27$

2. $r = \frac{a_4}{a_3} = \frac{-16-8i}{-4+8i} = \frac{-4-2i}{-1+2i} = \frac{(-4-2i)(-1-2i)}{(-1+2i)(-1-2i)} = \frac{4+10i-4}{1+4} = \frac{10i}{5} = 2i$

$a_1 = \frac{a_3}{r^2} = \frac{-4+8i}{-4} = 1-2i$

3. Since the powers of *i* repeat every 4 and $i + i^2 + i^3 + i^4 = 0$, $\sum_{k=1}^{22} 5i^k = 5\left(i^{21} + i^{22}\right) = 5(i-1)$

$\sum_{k=1}^{22} 2i^{3k} = 2\sum_{k=1}^{22}\left(\left(i^3\right)^k\right) = 2\sum_{k=1}^{22}\left((-i)^k\right)$ and since the powers of –*i* repeat every 4 and $(-i)^1 + (-i)^2 + (-i)^3 + (-i)^4 = 0$, this second sum = $2\left((-i)^{21} + (-i)^{22}\right) = 2(-i-1)$; $5(i-1) + 2(-i-1) = -7 + 3i$

Round 5

1. $y^2 - 16y + 6x + 79 = 0 \Rightarrow y^2 - 16y + 64 = -6x - 79 + 64 \Rightarrow (y-8)^2 = -6\left(x + 2\tfrac{1}{2}\right) \Rightarrow$ vertex = $\left(-2\tfrac{1}{2}, 8\right)$ and since the parabola opens left, the focus is p units left of vertex. $4p = 6 \Rightarrow p = 1\tfrac{1}{2} \Rightarrow \text{focus} = \left(-2\tfrac{1}{2} - 1\tfrac{1}{2}, 8\right) = (-4, 8)$

2. $x^2 + y^2 - 12x + 12y + 36 = 0 \Rightarrow x^2 - 12x + 36 + y^2 + 12y + 36 = 36 \Rightarrow$ $(x-6)^2 + (y+6)^2 = 36 \Rightarrow \text{center} = (6, -6)$ and $\text{radius} = 6$; $(15, -6)$ is 9 units from the center $\Rightarrow$ distance from $(15, -6)$ to the point of tangency = $\sqrt{9^2 - 6^2} = 3\sqrt{3^2 - 2^2} = 3\sqrt{5}$

3. The hyperbola has the same center $(-2, 4)$ as the ellipse; the vertices of the ellipse are 5 units up and down from the center $\Rightarrow$ c = 5 for the hyperbola; for the ellipse, $c^2 = a^2 - b^2 \Rightarrow c^2 = 25 - 16 = 9 \Rightarrow c = 3 \Rightarrow a = 3$ for the hyperbola; for the hyperbola, $c^2 = a^2 + b^2 \Rightarrow 25 = 9 + b^2 \Rightarrow b^2 = 16$; from these facts the equation of the hyperbola is $\dfrac{(y-4)^2}{9} - \dfrac{(x+2)^2}{16} = 1$; let x = 6: $\dfrac{(y-4)^2}{9} - \dfrac{(6+2)^2}{16} = 1 \Rightarrow \dfrac{(y-4)^2}{9} - 4 = 1 \Rightarrow$ $(y-4)^2 = 45 \Rightarrow y - 4 = \pm 3\sqrt{5} \Rightarrow y = 4 \pm 3\sqrt{5}$

Team Round

1.

Sum of pips	possibilities	permutations
3	1-1-1	1
4	1-1-2	3
15	5-5-5	1
	4-5-6	6
	6-6-3	3
16	4-6-6	3
	5-5-6	3
17	5-6-6	3
18	6-6-6	1

probability = $\dfrac{24}{6^3} = \dfrac{1}{9}$

2. Each of the four smaller circles have a radius = 2cm. The shaded area = area of circle O – 4 $\times$ area of smaller circle + area where circles A, B, C, and D overlap, which consists of 8 - 90° segments = $p \cdot 4^2 - 4 \cdot p \cdot 2^2 + 8\left(\tfrac{1}{4}p \cdot 2^2 - \tfrac{1}{2} 2 \cdot 2\right) = 16p - 16p + 8(p - 2) = 8p - 16$

3. probability = $\dfrac{\binom{4}{1}\binom{13}{3}\binom{39}{2} + \binom{4}{1}\binom{13}{4}\binom{39}{1} + \binom{4}{1}\binom{13}{5}}{\binom{52}{5}} \approx 0.3711$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2001

ROUND 1 – Volume and Surface Area of Solids

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE

1. A face diagonal of cube C is $2\sqrt{3}$ inches long. Find the number of cubic inches in the volume of cube D whose side has the same length as a main diagonal (not the face diagonal) of cube C.

2. A spherical orange is sliced into four congruent pieces. If the total surface area (plane and curved) of one piece of the orange is 32π cm^2, find the number of cubic centimeters in the volume of this one piece.

3. A regular pyramid with a square base has each of its lateral faces making a $60°$ angle with the plane of the square. If the total surface area of this pyramid is 36 m^2, find the number of cubic meters in its volume.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2001

ROUND 2 – Inequalities and Absolute Value

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all values of *x*, $x \in \Re$, satisfying the equation, $|2x-3| = 12 - |6-4x|$

2. Find all values of *x*, $x \in \Re$, satisfying the equation, $3x^2 + 8x - 4 = |3x+4|$

3. Find all values of *x*, $x \in \Re$, satisfying the inequality,
$\left\{ x \,\middle|\, \frac{x-4}{x^2-3x} \le 1 \right\}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2001

ROUND 3 – Similar Polygons, Circles and Areas Related to Circles

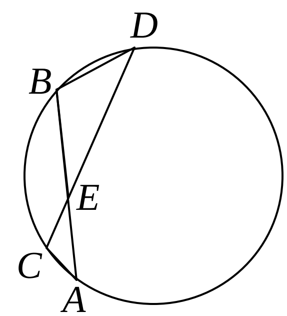
1. ____________________

2. ____________________

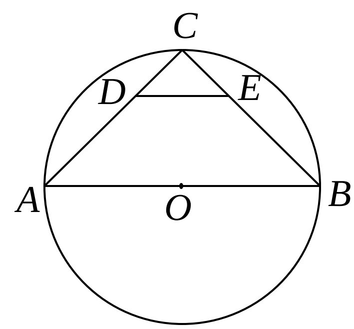
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

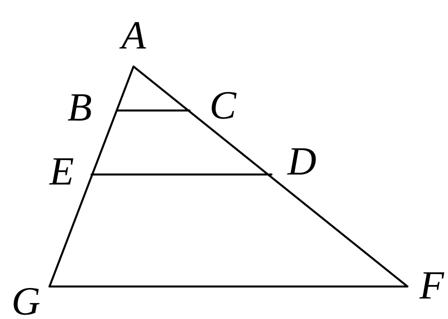
1. The circle on the right has chords $\overline{AB}$ and $\overline{CD}$ intersecting at point E. If $CE = 4$, $ED = 12$, and $BE = 8$, find the ratio of the area of ΔACE to the area of ΔBDE.



2. Given circle, center O, diameter $\overline{AB}$, isosceles $\Delta\ ACB$, $\overline{DE} \parallel \overline{AB}$, $\overline{ADC}$, $\overline{BEC}$, $AD{:}DC = 2{:}1$ and the circumference of circle O is 24π cm, find the number of square centimeters in the area of quadrilateral $ABED$.



3. In the diagram on the right, $\overline{BC} \parallel \overline{ED} \parallel \overline{GF}$, $BC : GF = 1 : 5$, and $AC : CD = 2 : 3$, find the ratio of the area of trapezoid $BCDE$ to the area of ΔAGF.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2001

ROUND 4 – Sequences and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the arithmetic sequence of complex numbers whose first term is $3+i$ and whose tenth term is $-15+28i$, find the sum of the first 20 terms of this sequence. Note $i=\sqrt{-1}$.

2. Find the following sum: $\sum_{k=1}^{165} \log_{10}\left(\frac{3k+2}{3k+5}\right)$

3. The sum of all the terms of an infinite geometric sequence is 512 and the second term of this sequence is 96, find all possible values for its first term.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2001

ROUND 5 – Conics

1. ________________________

2. $(\_\_\_,\_\_\_)$ $(\_\_\_,\_\_\_)$

3. $(\_\_\_\_,\_\_\_\_)$ ________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the conic, $\{(x, y) | 2x^2 - y^2 = 10\}$, find the equations of both asymptotes in slope-intercept form, which is $y = mx + b$.

2. Find the coordinates of the two points of intersection of line $\ell, \{(x, y) | x - 2y + 6 = 0\}$, with the parabola whose vertex is the origin and whose focus is the point $P\ (2, 0)$.

3. Given conic C, $\{(x, y) | 3x^2 + y^2 = 1\}$, a circle is drawn having the same center as conic C and containing its foci. Find in simplest form the coordinates of the point in the first quadrant where the circle intersects conic C.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2001

TEAM ROUND

3 pts. 1. ___________________

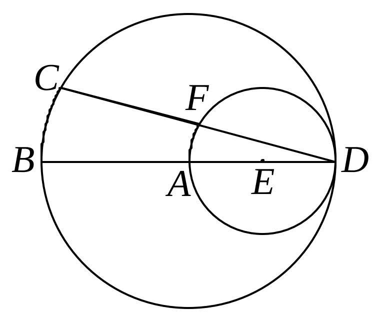
3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. In a box are 6 red, 5 blue, 4 green and 3 yellow marbles. If 4 marbles are drawn at random from the box, what is the probability that that there are not three or four matching in color? Express the probability either as a rational number in reduced form or if estimated, round off to four decimal places.

2. Given circles centered at points A and E such that circle E contains point A and is internally tangent to circle A at point D. If $\overline{BAED}, \overline{CFD}$, $m\angle D = 15°$, and $BD = 12$, find the area bounded by $\overline{AB}, \widehat{BC}, \overline{CF}$, and $\widehat{AF}$, as **boldly outlined** on the diagram. If estimating the area, round off the result to four decimal places.



3. An urn contains 6 red, 3 blue and 1 white marble. A regular decahedron has on its faces the numbers from 1 to 10, one number per face. In a game 2 marbles are picked at random from the urn and the decahedron is rolled. If both marbles are the same color and a prime number comes up on the top face, you win $20. If different colored marbles are picked and the number on the top face is not prime, you win $5. Otherwise, you win nothing. How many dollars is your expectation if you play one game?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2001

ANSWER SHEET:

ROUND 1

1. $54\sqrt{2}$ $\left(54\sqrt{2}\,in^3\right)$

2. $\frac{64p}{3}$ $\left(\frac{64p}{3}cm^3\right)$

3. 12 $(12m^3)$

ROUND 2

1. $-\frac{1}{2},\frac{7}{2}$ (– 0.5, 3.5)

2. $-\frac{11}{3},1$

3. $x<0$ or $x>3$ or $x=2$

ROUND 3

1. 1:4 $\left(\frac{1}{4}\right)$

2. 128 $\left(128cm^2\right)$

3. 21:100 $\left(\text{or } \frac{21}{100}\right)$

ROUND 4

1. $-320+590i$

2. – 2

3. 128, 384

ROUND 5

1. $y=\pm\sqrt{2}x$ $\left(y=\pm\sqrt{2}x+0\right)$

2. $(2,4),(18,12)$

3. $\left(\frac{\sqrt{6}}{6},\frac{\sqrt{2}}{2}\right)$

TEAM ROUND

3 pts. 1. $\frac{433}{510}\approx 0.8490$

3 pts. 2. $\frac{27+9p}{4}$ or equivalent. ≈ 13.8186

4 pts. 3. 5 $(\$5)$

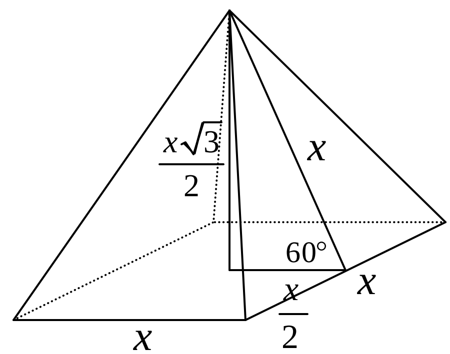
Detailed Solutions to GBML Meet 4, January 2001

Round 1

1. The length of the side of cube C = $\dfrac{2\sqrt{3}}{\sqrt{2}} = \sqrt{6}$ in. The diagonal of cube C = $\sqrt{6}\sqrt{3} = 3\sqrt{2}$ in. The volume of the second cube = $\left(3\sqrt{2}\right)^3 = 27\left(2\sqrt{2}\right) = 54\sqrt{2}$ cubic inches.

2. The total surface of the orange slice is made up of 2 semicircles and 1/4 the surface area of the sphere = $p\,r^2 + \frac{1}{4}\left(4p\,r^2\right) = 2pr^2 = 32p \rightarrow r = 4$ and the volume of the slice = $\frac{1}{4}\left(\frac{4}{3}p \cdot 4^3\right) = \frac{64p}{3}$

3. If x is one side of the square $\rightarrow$ x = slant height and $\frac{x\sqrt{3}}{2}$ is the height of the pyramid (see diagram.)



Total area = $x^2 + \frac{1}{2}(4x)x = 3x^2 = 36 \rightarrow x = 2\sqrt{3}$;

Volume = $\frac{1}{3}(12)(3) = 12cm^3$

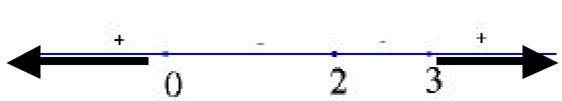
Round 2

1. $|2x-3| = 12 - |6-4x| \rightarrow |2x-3| = 12 - 2|3-2x| \rightarrow |2x-3| = 12 - 2|2x-3| \rightarrow 3|2x-3| = 12$ $\rightarrow |2x-3| = 4 \rightarrow 2x-3 = \pm 4 \rightarrow x = -\frac{1}{2}, \frac{7}{2}$

2. $3x^2 + 8x - 4 = |3x+4| \rightarrow$ case (i): $x \ge -\frac{4}{3}: 3x^2 + 8x - 4 = 3x + 4 \rightarrow 3x^2 + 5x - 8 = 0 \rightarrow$ $(3x+8)(x-1) = 0 \rightarrow x = 1$; case(ii): $x < -\frac{4}{3}: 3x^2 + 8x - 4 = -3x - 4 \rightarrow 3x^2 + 11x = 0 \rightarrow$ $x(3x+11) = 0 \rightarrow x = -\frac{11}{3}$; the 2 solutions are $-\frac{11}{3}, 1$

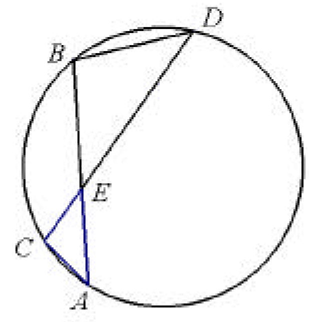
3. $\dfrac{x-4}{x^2-3x} \le 1 \rightarrow \dfrac{x-4-x^2+3x}{x^2-3x} \le 0 \rightarrow \dfrac{-x^2+4x-4}{x(x-3)} \le 0 \rightarrow \dfrac{x^2-4x+4}{x(x-3)} \ge 0 \rightarrow \dfrac{(x-2)^2}{x(x-3)} \ge 0$ $\rightarrow$ key values for x are 0 and 3 (excluded) and 2 (included); considering the value of the rational expression for each section of the number line reaches the following conclusion:

$$x < 0 \text{ or } x > 3 \text{ or } x = 2$$



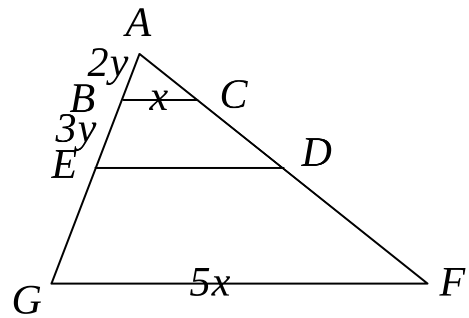
Round 3

1. $CE \cdot ED = AE \cdot ED \rightarrow AE = 6$; $\Delta AEC \sim \Delta DEB \rightarrow$ ratio of their areas = $\left(\frac{AE}{DE}\right)^2 = \left(\frac{6}{12}\right)^2 = 1:4$



2. $\Delta\ ABC$ is right isosceles; $C = 24\pi \rightarrow d = AB = 24 \rightarrow$ $AC = BC = 12\sqrt{2} \rightarrow CD = CE = 4\sqrt{2}$;
area of ABED = area of Δ ABC – area of Δ DEC =
$\frac{1}{2}\left(12\sqrt{2}\right)^2 - \frac{1}{2}\left(4\sqrt{2}\right)^2 = 144 - 16 = 128\ \text{cm}^2$

3. $\Delta ABC \sim \Delta AGF \rightarrow \frac{AB}{AG} = \frac{BC}{GF} \rightarrow \frac{2y}{AG} = \frac{1}{5} \rightarrow AG = 10y$
$\rightarrow$ height of trapezoid $BCDE$: height of ΔAGF = 3:10; call the respective heights $3h$ and $10h$.
$\Delta AED \sim \Delta AGF \rightarrow \frac{AE}{AG} = \frac{ED}{GF} \rightarrow \frac{5y}{10y} = \frac{ED}{5x} \rightarrow$



$ED = 2.5x$. Area of trapezoid $BCDE$ =
$\frac{1}{2}3h\left(x + 2.5x\right) = 5.25hx$; area of ΔAGF = $\frac{1}{2}(5x)(10h) = 25xh$;
ratio of the areas = $\frac{5.25hx}{25hx} = \frac{21}{100}$

Round 4

1. $-15 + 28i = 3 + i + 9d \rightarrow d = -2 + 3i$;
$S_{20} = \frac{20}{2}\left(2(3+i) + 19(-2+3i)\right) = 10(6 + 2i - 38 + 57i) = -320 + 590i$

2. $\sum_{k=1}^{165} \log_{10}\left(\frac{3k+2}{3k+5}\right) = \sum_{k=1}^{165}\left(\log_{10}(3k+2) - \log_{10}(3k+5)\right)$; since $3(k+1)+2 = 3k+5$, the second term being subtracted = 3rd term being added, and so on. This means all the terms add to 0 except the first and last terms. The sum =
$\log_{10}(3(1)+2) - \log_{10}(3(165)+5) = \log_{10}5 - \log_{10}500 = \log_{10}\left(\frac{5}{500}\right) = \log_{10}\left(\frac{1}{100}\right) = -2$

3. $\frac{a}{1-r} = 512$ and $ar = 96 \rightarrow \frac{96}{r} = 512(1-r) \rightarrow 3 = 16r(1-r) \rightarrow 16r^2 - 16r + 3 = 0$
$\rightarrow (4r-1)(4r-3) = 0 \rightarrow r = \frac{1}{4}, \frac{3}{4} \rightarrow a = 96\left(\frac{4}{3}\right), 96\left(\frac{4}{1}\right) = 128,\ 384$

Round 5

1. $2x^2 - y^2 = 10 \rightarrow \frac{x^2}{5} - \frac{y^2}{10} = 1 \rightarrow$ asymptotes are $\frac{x^2}{5} = \frac{y^2}{10} \rightarrow y^2 = 2x^2 \rightarrow y = \pm\sqrt{2}x$

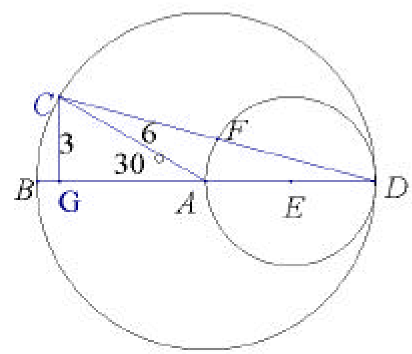
2. The equation of the parabola is $y^2 = 8x$ and since $x = 2y - 6 \rightarrow y^2 = 8(2y-6) \rightarrow$ $y^2 - 16y + 48 = 0 \rightarrow (y-4)(y-12) = 0 \rightarrow y = 4{,}12 \rightarrow x = 2{,}18$ respectively $\rightarrow$ points of intersection are $(2,4),(18,12)$

3. $3x^2 + y^2 = 1 \rightarrow \frac{x^2}{1/3} + \frac{y^2}{1} = 1 \rightarrow c^2 = 1 - 1/3 = 2/3 \rightarrow$ equation of the circle is $x^2 + y^2 = \frac{2}{3} \rightarrow$ subtracting this from the original equation, $2x^2 = \frac{1}{3} \rightarrow x^2 = \frac{1}{6} \rightarrow y^2 = \frac{1}{2}$ since the point of intersection is in quadrant I $\rightarrow$ point = $\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}\right)$

Team Round

1. Number of elements in the sample space = $\binom{18}{4} = 3060$; event having 3 of 1 color = $\binom{6}{3}\binom{12}{1} + \binom{5}{3}\binom{13}{1} + \binom{4}{3}\binom{14}{1} + \binom{3}{3}\binom{15}{1} = 441$; event having 4 of 1 color = $\binom{6}{4} + \binom{5}{4} + \binom{4}{4} = 21$; probability neither event occurs = $\frac{3060 - 441 - 21}{3060} = \frac{433}{510} \approx 0.8490$

2. $m\angle BAC = 30° \rightarrow CG = 3$; area of sector ABC = $\frac{1}{12}(36p) = 3p$; area of $\Delta ACD = \frac{1}{2}(6)(3) = 9 \rightarrow$ area bounded by $\overline{DB}, \overline{DC}, \widehat{BC} = 3p + 9$; by similarity area bounded by $\overline{DA}, \overline{DF}, \widehat{AF} = \frac{1}{4}(3p + 9)$; by subtraction, the area bounded by $\overline{AB}, \overline{CF}, \widehat{AF}, \widehat{BC} = \frac{3}{4}(3p + 9)$



3. Probability 2 marbles of same color = $\frac{{}_6C_2 + {}_3C_2}{{}_{10}C_2} = \frac{2}{5} \rightarrow$ probability 2 marbles of different colors = $\frac{3}{5}$; probability of a prime = $\frac{2}{5} \rightarrow$ probability a non-prime = $\frac{3}{5}$; expectation = $\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)(20) + \left(\frac{3}{5}\right)\left(\frac{3}{5}\right)(5) = 5$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2002

ROUND 1 – Volume and Surface Area of Solids

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The volume of a cube is 64 cubic inches. Let the length of a diagonal of this cube divided by the length of its face diagonal equal P. Find the number of square inches in the total surface area of a cube with side of length P inches.

2. Find the number of square centimeters in the lateral area of a regular hexagonal pyramid with a base perimeter of 36 cm and a height of 3 cm.

3. A right circular cylinder has a radius of 6 in. and a height of 30 in. Its total surface area is equal to the total surface area of a hemisphere. If this hemisphere is filled to half of its capacity with water and this water is then poured into the empty cylinder, what would be the number of inches in the depth of the water in the cylinder?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2002

ROUND 2 – Inequalities and Absolute Value

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all values of *x*, $x \in \Re$, satisfying the inequality, $\left\{ x \,\middle|\, \frac{x}{|x|-2} > 0 \right\}$

2. Find all values of *x*, $x \in \Re$, satisfying the equation, $\left\{ x \,\middle|\, |2x-3| = |x+6| + 3 \right\}$

3. Find all values of *x*, $x \in \Re$, satisfying the inequality,
$$\left\{ x \,\middle|\, \frac{4}{x+6} - \frac{4}{2-x} \geq \frac{x^2}{12-4x-x^2} \right\}$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2002

ROUND 3 – Similar Polygons, Circles and Areas Related to Circles

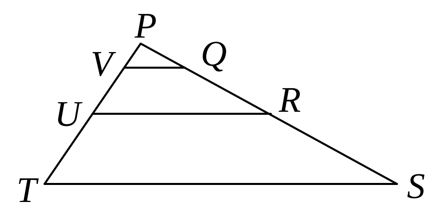
1. ____________________

2. ____________________

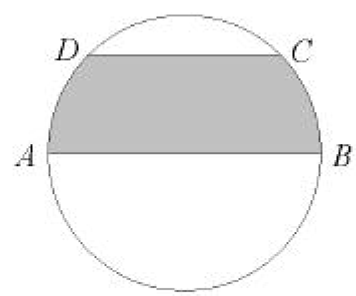
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

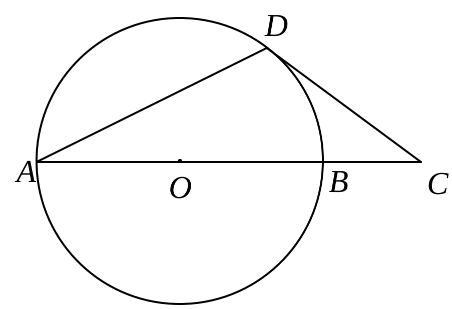
1. Given $\overline{PQRS}$, $\overline{PVUT}$, $\overline{QV} \parallel \overline{RU} \parallel \overline{ST}$, and $PQ:QR:RS = 1:2:3$, find the ratio of the area of ΔPQV to the area of trapezoid *RSTU*.



2. Given $\overline{AB}$ is a diameter of the circle to the right, $AB = 20$, and $m\overset{\frown}{AD} = m\overset{\frown}{BC} = 45°$, find the area of the shaded region.



3. Given circle centered at *O*, $\overline{AOBC}$, $\overline{CD}$ tangent to the circle at point *D*, $BC = 6$, and $CD = 12$, find the area of ΔACD.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2002

ROUND 4 – Sequences and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given a geometric sequence in which $a_1 = \frac{1}{8}$, a_3 is a real number, and $a_4 = -i$, find a_{10}. Note $i = \sqrt{-1}$.

2. Given the series, $-47 - 39 - 31 - 23 - 15 - \ldots$, what is the least number of terms necessary for the sum to be greater than 200?

3. The three terms, $x, 5x+1$, and y, form an arithmetic sequence. If 2 is added to the first term, 3 is subtracted from the second term, and 4 is subtracted from the third term, the sequence is now geometric. Find all values for x which will make this true.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2002

ROUND 5 – Conics

1. $(\_\_\_,\_\_\_)$

2. _________________________

3. _________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the conic C, $\{(x,y) \mid y^2+4y+12x-32=0\}$, find the coordinates of its focus.

2. A circle contains the point $P(2,9)$ and is tangent to both axes. Find all possible values for the radius of this circle.

3. The conic C, $\{(x,y) \mid 12x^2-4y^2-72x-24y+24=0\}$, has a focus F in quadrant III with coordinates (a,b). If point $P(a,d)$ lies on conic C, find all possible values for d.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2002

TEAM ROUND (<u>12 MINUTES LONG</u>)

3 pts. 1. ___________________

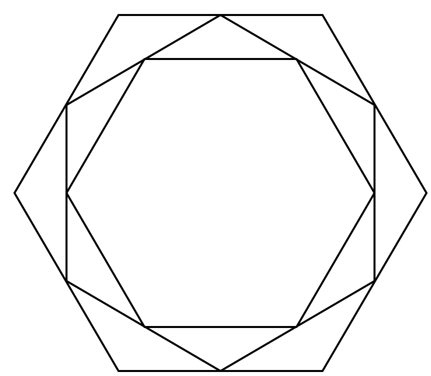
3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. From a box containing 10 red, 8 white, and 7 blue marbles, 6 are chosen at random. What is the probability that exactly 4 are the same color? Express the result in reduced rational form or if estimated round off to exactly 4 decimal places.

2. The midpoints of the sides of a regular hexagon are connected forming a second regular hexagon. Then the midpoints of the sides of this second hexagon are connected forming a third regular hexagon. (See the figure to the right.) If this process continues forever, the sum of the areas of <u>all</u> the hexagons equals $\sqrt{3}$ square centimeters. Find the exact number of centimeters (simplest radical form) in the sum of the perimeters of <u>all</u> the hexagons.



3. How many different 4-letter permutations are possible using any of the letters in the word ***MINIMUM***.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2002

ANSWER SHEET:

ROUND 1

1. 9 (9 square inches)
2. 108 ($108\ \text{cm}^2$)
3. 16 (16 inches)

ROUND 2

1. $-2 < x < 0$ or $x > 2$
2. -2, 12
3. $x < -6$ or $x > 2$ or $x = -4$

ROUND 3

1. 1:27 $\left(\text{or } \frac{1}{27}\right)$
2. $50 + 25p$
3. $\frac{432}{5}$ (or $86\frac{2}{5}$ or 86.4)

ROUND 4

1. $64i$
2. 16
3. $\frac{1}{4}$, 2

ROUND 5

1. $(0, -2)$
2. 5, 17
3. -9, 3

TEAM ROUND

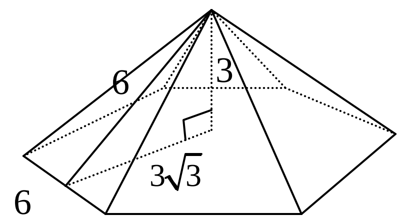
3 pts. 1. $\frac{211}{1012} \approx 0.2085$

3 pts. 2. $4\sqrt{6} + 6\sqrt{2}$

4 pts. 3. 114

ROUND 1 – Volume and Surface Area of Solids

1. The volume of the original cube is irrelevant. If s = length of its side, then $s\sqrt{3}=$ its diagonal's length and $s\sqrt{2}=$ length of the diagonal of its face $\Rightarrow \dfrac{\sqrt{3}}{\sqrt{2}}=P$. The surface area of the 2nd cube = $6P^2=6\left(\sqrt{3}/\sqrt{2}\right)^2=6\cdot 3/2=9$ square inches.

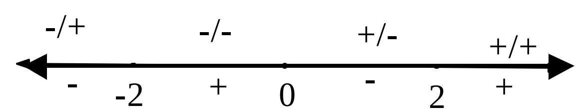


2. The side of the regular hexagon = 6 cm $\Rightarrow$ apothem of the regular hexagon = $3\sqrt{3}$ cm $\Rightarrow$ slant height = 6 cm. The lateral area = $\dfrac{1}{2}P\cdot l=\dfrac{1}{2}\cdot 36\cdot 6=108\,\text{cm}^2$.

3. The total surface area of the cylinder = $2\pi r(r+h)=12\pi\cdot 36$. If R = radius of the hemisphere $\Rightarrow 3\pi R^2=12\pi\cdot 36\Rightarrow R^2=144\Rightarrow R=12$. Half the volume of the hemisphere = $\dfrac{1}{2}\cdot\dfrac{2}{3}\pi\cdot 12^3=12\cdot 12\cdot 4\pi$ in.^{3}. To find the height of the water in the cylinder divide this by 36π in.$^2\Rightarrow$ height of the water $=\dfrac{12\cdot 12\cdot 4\pi}{36\pi}=16$in.

ROUND 2 – Inequalities and Absolute Value

1. To solve $\dfrac{x}{|x|-2}>0$, identify the key values for x, which are –2, 0, 2 (values that make the numerator and denominator 0). Now section off the number line (see below): Therefore the solution is $-2<x<0$ or $x>2$.



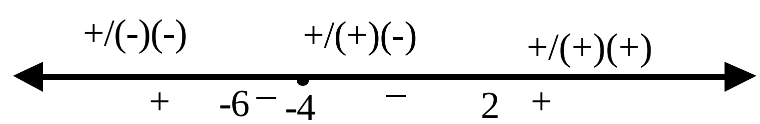
2. To solve $|2x-3|=|x+6|+3$, consider the key values for x (values that make each absolute value expression = 0) which are –6 and 1.5. Now consider three cases:
(i) $x\ge 1.5$: $2x-3=x+6+3\Rightarrow x=12$, which satisfies the restriction on x.
(ii) $-6\le x\le 1.5$: $3-2x=x+6+3\Rightarrow x=-2$, which satisfies the restriction on x.
(iii) $x\le -6$: $3-2x=-x-6+3\Rightarrow x=6$, which does not satisfy the restriction on x.
Therefore the solutions for x are –2, 12 only.

3. $\dfrac{4}{x+6}-\dfrac{4}{2-x}\ge\dfrac{x^2}{12-4x-x^2}\Rightarrow\dfrac{4}{x+6}+\dfrac{4}{x-2}\ge\dfrac{-x^2}{x^2+4x-12}\Rightarrow$

$$\frac{x^2}{x^2+4x-12}+\frac{4}{x+6}+\frac{4}{x-2}\ge 0\Rightarrow\frac{x^2}{(x+6)(x-2)}+\frac{4(x-2)}{(x+6)(x-2)}+\frac{4(x+6)}{(x-2)(x+6)}\ge 0$$

$$\Rightarrow\frac{x^2+4x-8+4x+24}{(x+6)(x-2)}\ge 0\Rightarrow\frac{x^2+8x+16}{(x+6)(x-2)}\ge 0\Rightarrow\frac{(x+4)^2}{(x+6)(x-2)}\ge 0.$$

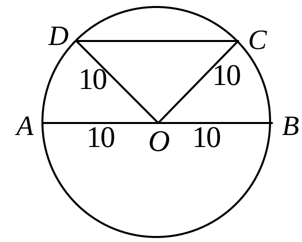
Key values: –6, 2(excluded) and –4 (included). Now, section off the number line $\Rightarrow x<-6$ or $x>2$ or $x=-4$.



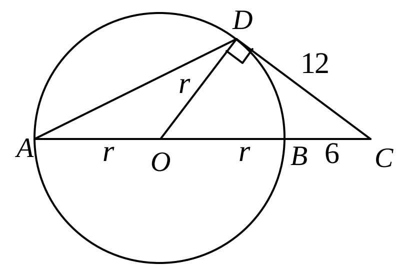
ROUND 3 – Similar Polygons, Circles and Areas Related to Circles

1. Since $PQ:QR:RS = 1:2:3 \Rightarrow PQ:PR:PS = 1:3:6 \Rightarrow$ area of ΔPQV: area of ΔPRU: area of ΔPST = 1:9:36 $\Rightarrow$ area of ΔPQV: area of trapezoid $RSTU$ $= 1:36-9 = 1:27$.

2. Let O be the center of the circle. Draw radii $\overline{OC}$ and $\overline{OD}$. The shaded area consists of right ΔCOD and two 45° sectors. Therefore the area = $\frac{1}{2}(10)(10)+\frac{90}{360}(10)^2 p = 50+25p$.



3. Draw radius $\overline{OD}$. Call its length r. $CD^2 = CA \cdot CB \Rightarrow$ $12^2 = 6(2r+6) \Rightarrow 2r+6 = 24 \Rightarrow r = 9$. To find the area of ΔACD, you need the length of the altitude from D. Since ΔCOD is right $\Rightarrow$ if h = altitude, then $\frac{15h}{2} = \frac{12 \cdot 9}{2} \Rightarrow h = \frac{36}{5}$.
Therefore the area of ΔACD = $\frac{1}{2}(24)\left(\frac{36}{5}\right) = \frac{432}{5}$.



ROUND 4 – Sequences and Complex Numbers

1. $a_4 = a_1 \cdot r^3 \Rightarrow -i = \frac{1}{8} r^3 \Rightarrow r^3 = -8i$. Since a_3 is real the only possible value for r = $2i \Rightarrow$ $a_{10} = a_4 \cdot r^6 = -i(-2i)^6 = -i(-64) = 64i$.

2. This is an arithmetic series with $d = 8$. The sum of the first n terms = $\frac{n}{2}(2(-47)+(n-1)(8)) = \frac{n}{2}(8n-102) = n(4n-51)$. You want the smallest value of n such that $n(4n-51) > 200$. You could use the quadratic formula to find what value of n makes the sides equal, but trial and error is quicker and less complicated.
If $n = 15 \Rightarrow n(4n-51) = 15 \cdot 9 < 200$. If $n = 16 \Rightarrow n(4n-51) = 16 \cdot 13 > 200$. Therefore the answer is 16.

3. Since the original terms are arithmetic, then $y + x = 10x+2 \Rightarrow y = 9x+2$. The new terms, $x+2$, $5x-2$, and $y-4$ are geometric, therefore $(5x-2)^2 = (x+2)(y-4)$
$\Rightarrow 25x^2 - 20x + 4 = (x+2)(9x-2) = 9x^2 + 16x - 4 \Rightarrow 16x^2 - 36x + 8 = 0 \Rightarrow$
$4x^2 - 9x + 2 = 0 \Rightarrow (4x-1)(x-2) = 0 \Rightarrow x = \frac{1}{4}$, 2.

ROUND 5 – Conics

1. $y^2+4y+12x-32=0 \Rightarrow y^2+4y+4=-12x+36 \Rightarrow (y+2)^2=-12(x-3) \Rightarrow$ vertex = $(3,-2)$. This parabola "opens" to the left and since $4p=12 \Rightarrow p=3 \Rightarrow$ focus is 3 units to the left of the vertex $\Rightarrow$ focus = $(3-3,-2)=(0,-2)$.

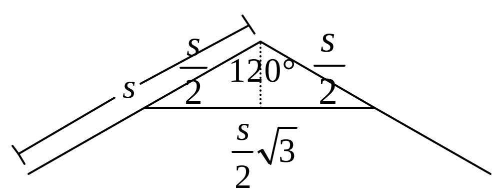
2. Since the circle is tangent to both axes, its center must lie on the line $y=x$.
Let $O(h,h)$ be its center $\Rightarrow (2-h)^2+(9-h)^2=h^2 \Rightarrow h^2-4h+4-18h+81=0$
$\Rightarrow h^2-22h+85=0 \Rightarrow (h-5)(h-17)=0 \Rightarrow h$ = 5, 17.

3. $12x^2-4y^2-72x-24y+24=0 \Rightarrow 12(x^2-6x+9)-4(y^2+6y+9)=-24+108-36 \Rightarrow$
$12(x-3)^2-4(y+3)^2=48 \Rightarrow \dfrac{(x-3)^2}{4}-\dfrac{(y+3)^2}{12}=1 \Rightarrow$ center of hyperbola = $(3,-3)$;
foci are on the same horizontal line as the center and $c^2=4+12 \Rightarrow c=4 \Rightarrow$ foci are
$(3\pm 4,-3) \Rightarrow$ focus in quadrant III is $(-1,-3)$; when $x=-1 \Rightarrow \dfrac{(-1-3)^2}{4}-\dfrac{(y+3)^2}{12}=1$
$\Rightarrow 4-\dfrac{(y+3)^2}{12}=1 \Rightarrow \dfrac{(y+3)^2}{12}=3 \Rightarrow (y+3)^2=36 \Rightarrow y=-3\pm 6=-9, 3.$

TEAM ROUND

1. The sample space has ${}_{25}C_6$ elements in it. The successful events are choosing 4 reds and 2 non-reds, 4 whites and 2 non-whites, 4 blues and 2 non-blues.
Therefore the probability = $\dfrac{{}_{10}C_4\cdot{}_{15}C_2+{}_{8}C_4\cdot{}_{17}C_2+{}_{7}C_4\cdot{}_{18}C_2}{{}_{25}C_6}=\dfrac{211}{1012}\approx 0.2085$

2. The ratio of the sides of each hexagon to the previous one =
$\dfrac{\sqrt{3}}{2}$ (See the figure on the right.) $\Rightarrow$ ratio of perimeters
= $\dfrac{\sqrt{3}}{2}$ and the ratio of area = $\left(\dfrac{\sqrt{3}}{2}\right)^2=\dfrac{3}{4}$; since the sum



of areas = $\sqrt{3}$ and if s = side of the first, then $\dfrac{\frac{3}{2}s^2\sqrt{3}}{1-\frac{3}{4}}=\sqrt{3} \Rightarrow 6s^2\sqrt{3}=\sqrt{3} \Rightarrow s^2=\dfrac{1}{6} \Rightarrow s=\dfrac{\sqrt{6}}{6}$
$\Rightarrow$ P $=\sqrt{6}$ $\Rightarrow$ sum of perimeters = $\dfrac{\sqrt{6}}{1-\frac{\sqrt{3}}{2}}=\dfrac{2\sqrt{6}}{2-\sqrt{3}}=2\sqrt{6}\left(2+\sqrt{3}\right)=4\sqrt{6}+6\sqrt{2}$.

3. Consider 5 cases: (i) *MMM* and a 4th letter (ii) *MMII* (iii) *MM* and 2 different letters (iv) *II* and 2 different letters (same number as (iii)) (v) 4 different letters;
therefore the number of permutations = $3\cdot\dfrac{4!}{3!}+\dfrac{4!}{2!2!}+3\cdot\dfrac{4!}{2!}+3\cdot\dfrac{4!}{2!}+4!=$ 114.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2003

ROUND 1 – Volume and Surface Area of Solids

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A cylinder with a radius of 4 cm and a height of 10 cm is made of solid mathelite (an unusual substance) and weighs 48 grams. How many grams would a cone of radius 10 cm and height of 3 cm also made of solid mathelite weigh?

2. A sphere and a cylinder have the same radius and volume. Find the ratio of the surface area of the sphere to the total surface area of the cylinder.

3. Given a regular triangular pyramid with the length of each side of its base measuring 12cm and its height measuring $\sqrt{13}$ cm. A second regular triangular pyramid is constructed by connecting the midpoints of the sides of the base of the first pyramid with the same height. Find the number of square centimeters in the difference of the lateral areas of the two pyramids.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2003

ROUND 2 – Inequalities and Absolute Value

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all values of x, $x \in \Re$, satisfying the inequality,

$\{x \text{ such that } 5x^3 < 2x^2 - 15x + 6\}$.

2. Find all values of x, $x \in \Re$, satisfying the equation,

$\{x \text{ such that } 2|3x-1| + 3|2x+5| = 17\}$.

3. Find the <u>number of integers</u> satisfying the inequality, $\left|\dfrac{3x^2 - 3x - 6}{4x+4}\right| \le 7$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2003

ROUND 3 – Similar Polygons, Circles and Areas Related to Circles

1. ____________________

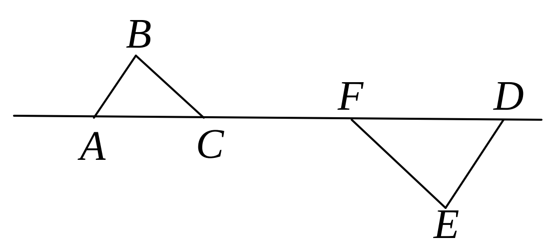
2. ____________________

3. ____________________

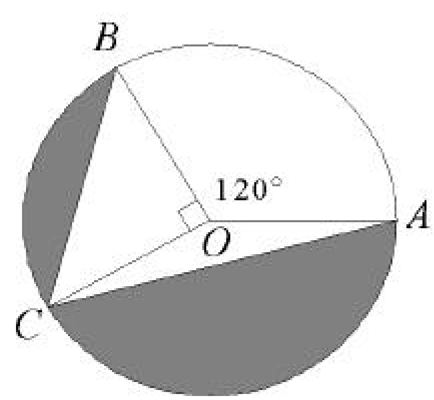
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. One regular hexagon has an area of $27\sqrt{3}\ \text{cm}^2$. A second regular hexagon has its longest diagonal with length equaling 12cm. Find the ratio of the perimeters of the smaller to the larger regular hexagon.

2. Given $\overline{AB}$ parallel to $\overline{DE}$, $\overline{BC}$ parallel to $\overline{EF}$, $AB = 4$, $DE = 5.5$, $AF = 16$, and $CD = 19$, find the length of $\overline{CF}$.



3. Given circle with center O, $m\angle AOB = 120°$, $\overline{OB} \perp \overline{OC}$, and the area of sector AOB = $4p$, find the shaded area in the figure on the right.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2003

ROUND 4 – Sequences and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

Note in this round, $i=\sqrt{-1}$.

1. Given a sequence in which $a_1 = 2$, and $a_n = \begin{cases} a_{n-1} + i, \text{ if } n \text{ is even.} \\ ia_{n-1}, \text{ if } n \text{ is odd.} \end{cases}$, find the next term which is a real number.

2. The sum of first n terms of the arithmetic sequence, $-13+64i, -10+60i, -7+56i, \ldots$, is the real number S. Find the value of S.

3. The sum of an infinite geometric series is 20 and the sum of its first three terms is $\frac{185}{16}$. Find the first term of this series.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2003

ROUND 5 – Conics

1. $(\_\_\_\_,\_\_\_\_)$

2. ________________________

3. $(\_\_\_\_,\_\_\_\_)$

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A diameter of a circle has endpoints which are the focus and vertex of the conic, $\left\{(x,y)\middle|(x+4)^2=-8(y-3)\right\}$. Find the coordinates of the center of this circle.

2. Given the conic C, $\left\{(x,y)\middle|9x^2-y^2+54x-20y-55=0\right\}$, find the distance between the foci of C.

3. Given conic E, $\left\{(x,y)\middle|10x^2+y^2-30=0\right\}$, d equals the shortest distance from any point on E to the center of E. Point $P(m,n)$ is on E and is in the first quadrant. The distance from P to the center of E equals $2d$. Find the ordered pair (m,n).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2003

TEAM ROUND: Time limit: 12 minutes

3 pts. 1. ___________________

3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Right circular cone C has a base with radius18cm and a height of 24cm. Sphere S having a volume of $288p$ cm^3 is dropped into cone C (vertex down). Sphere T can fit into the space in cone C and under sphere S. Find the number of square centimeters in the largest possible surface area for sphere T.

2. Al, Bill, and Carol play a 3-way game that always ends with only one winner. Carol is three times as likely to win a game as Al, who is twice as likely to win a game as Bill. If they play four games, what is the probability that Carol wins 2 of them and Al wins the other two? Express the answer as a reduced rational number.

3. How many different 5-letter permutations are possible using any of the letters in the word ***REPEATED*** if the first and last letters are consonants?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2003

ANSWER SHEET:

ROUND 1

1. 30 (30g)
2. $\frac{6}{7}$
3. 54 (54cm^2)

ROUND 2

1. $x < \frac{2}{5}$ (or $x < 0.4$)
2. $-\frac{5}{2} \le x \le \frac{1}{3}$ or equivalent
3. 18

ROUND 3

1. $\frac{\sqrt{2}}{2}$ (or $\sqrt{2} : 2$)
2. 8
3. $8p - 9$

ROUND 4

1. – 3
2. 1155
3. 5

ROUND 5

1. $(-4, 2)$
2. $4\sqrt{10}$
3. $\left(\sqrt{2}, \sqrt{10}\right)$

TEAM ROUND

3 pts. 1. $9p$

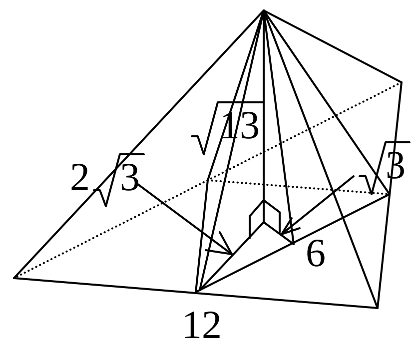
3 pts. 2. $\frac{32}{243}$

4 pts. 3. 408

ROUND 1 – Volume and Surface Area of Solids

1. The volume of the cylinder = $p(4)^2(16) = 160p$; volume of the cone = $\frac{1}{3}p(10)^2(3) = 100p$; the ratio of their volumes = ratio of their weights; let w = weight of the cone $\Rightarrow \frac{160p}{100p} = \frac{48}{w} \Rightarrow \frac{8}{5} = \frac{48}{w} \Rightarrow w = 30$.

2. Let the radius of the sphere = radius of the cylinder = r; let the height of the cylinder = h; $\frac{4}{3}pr^3 = pr^2h \Rightarrow h = \frac{4}{3}r \Rightarrow \frac{4pr^2}{2pr(r+h)} = \frac{2r}{r+\frac{4}{3}r} = \frac{6}{7}$.



3. The apothem of the equilateral Δ of side 12 = $2\sqrt{3}$ $\Rightarrow$ slant height of the first pyramid = $\sqrt{\left(\sqrt{13}\right)^2 + \left(2\sqrt{3}\right)^2} = 5$; the 2nd pyramid has an equilateral Δ of side 6 $\Rightarrow$ the apothem = $\sqrt{3} \Rightarrow$ slant height of the 2nd pyramid = $\sqrt{\left(\sqrt{13}\right)^2 + \left(\sqrt{3}\right)^2} = 4$; the difference of the lateral areas = $\frac{1}{2}(36)(5) - \frac{1}{2}(18)(4) = 54$.

ROUND 2 – Inequalities and Absolute Value

1. $5x^3 < 2x^2 - 15x + 6 \Rightarrow 5x^3 - 2x^2 + 15x - 6 < 0 \Rightarrow x^2(5x-2) + 3(5x-2) < 0 \Rightarrow (x^2+3)(5x-2) < 0 \Rightarrow 5x - 2 < 0 \Rightarrow x < 0.4$

2. To solve $2|3x-1| + 3|2x+5| = 17$, consider the key values for x (values that make each absolute value expression = 0) which are $\frac{1}{3}$ and $-\frac{5}{2}$. Now consider three cases:

(i) $x \geq \frac{1}{3}$: $2(3x-1) + 3(2x+5) = 17 \Rightarrow 12x + 13 = 17 \Rightarrow x = \frac{1}{3}$, which satisfies the restriction on x. (ii) $-\frac{5}{2} \leq x \leq \frac{1}{3}$: $-2(3x-1) + 3(2x+5) = 17 \Rightarrow 17 = 17 \Rightarrow -\frac{5}{2} \leq x \leq \frac{1}{3}$

(iii) $x \leq -\frac{5}{2}$: $-2(3x-1) - 3(2x+5) = 17 \Rightarrow -12x = 30 \Rightarrow x = -\frac{5}{2}$, which satisfies the restriction on x. Therefore the solution is $-\frac{5}{2} \leq x \leq \frac{1}{3}$

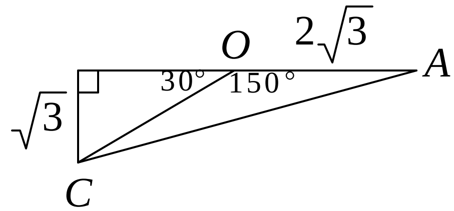
3. $\left|\frac{3x^2 - 3x - 6}{4x+4}\right| \leq 7 \Rightarrow \left|\frac{3(x+1)(x-2)}{4(x+1)}\right| \leq 7 \Rightarrow x \neq -1 \, and \, |x-2| \leq \frac{28}{3} \Rightarrow$ $-9\tfrac{1}{3} \leq x - 2 \leq 9\tfrac{1}{3} \Rightarrow -7\tfrac{1}{3} \leq x \leq 11\tfrac{1}{3}$ and $x \neq -1 \Rightarrow x$ has 18 integer values.

ROUND 3 – Similar Polygons, Circles and Areas Related to Circles

1. Since the area of the first regular hexagon is $27\sqrt{3} \Rightarrow \dfrac{3s^2\sqrt{3}}{2} = 27\sqrt{3} \Rightarrow s^2 = 18 \Rightarrow$ $s = 3\sqrt{2}$; since the longest diagonal of the 2nd hexagon = 12 $\Rightarrow$ its side = 6; the ratio of the perimeters, smaller to larger, = ratio of their sides = $\dfrac{3\sqrt{2}}{6} = \dfrac{\sqrt{2}}{2}$

2. Let $CF = x \Rightarrow AC = 16 - x$ and $DF = 19 - x$; $\Delta ABC \sim \Delta DEF$ by parallel line theorems $\Rightarrow$ $\dfrac{4}{16-x} = \dfrac{5.5}{19-x} \Rightarrow 76 - 4x = 88 - 5.5x \Rightarrow 1.5x = 12 \Rightarrow x = 8.$

3. Since the area of the 120° sector = $4p \Rightarrow$ area of circle O = $12p \Rightarrow r^2 = 12 \Rightarrow r = 2\sqrt{3}$. The shaded area = 90° segment + 150° segment; the area of the 90° segment $= \dfrac{1}{4}(12p) - \dfrac{1}{2}\left(2\sqrt{3}\right)^2 = 3p - 6$;



the area of the 150° segment = $\dfrac{5}{12}(12p) - \dfrac{1}{2}\left(2\sqrt{3}\right)\left(\sqrt{3}\right) =$ (See the diagram above.) $5p - 3$; the total area = $8p - 9$

ROUND 4 – Sequences and Complex Numbers

1. $a_1 = 2 \Rightarrow a_2 = 2 + i \Rightarrow a_3 = 2i - 1 \Rightarrow a_4 = 3i - 1 \Rightarrow a_5 = -3 - i \Rightarrow a_6 = -3.$ [Note you are alternating between adding i and then multiplying by i as you go from one term to the next.]

2. This is an arithmetic sequence in which $d = 3 - 4i$. The sum of the first n terms = $\dfrac{n}{2}\left(2(-13 + 64i) + (n-1)(3 - 4i)\right)$. Since this sum is a real number $\Rightarrow$ $2(64i) + (n-1)(-4i) = 0 \Rightarrow n - 1 = 32 \Rightarrow n = 33 \Rightarrow$ sum = $\dfrac{33}{2}\left(2(-13) + 32(3)\right) = 33 \cdot 35 = 1155.$

3. Let a = the first term and let r = ratio between terms $\Rightarrow \dfrac{a}{1-r} = 20$ and $\dfrac{a\left(1-r^3\right)}{1-r} = \dfrac{185}{16}$; $20\left(1 - r^3\right) = \dfrac{185}{16} \Rightarrow 1 - r^3 = \dfrac{37}{64} \Rightarrow r^3 = \dfrac{27}{64} \Rightarrow r = \dfrac{3}{4} \Rightarrow \dfrac{a}{1 - 3/4} = 20 \Rightarrow a = 5.$

ROUND 5 – Conics

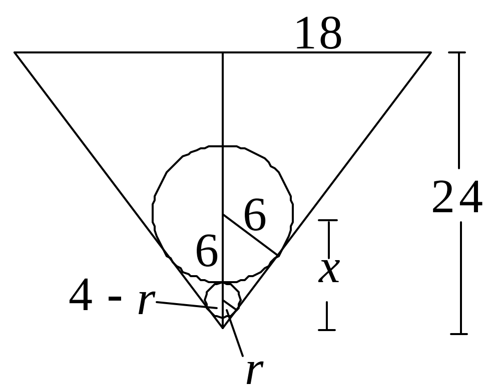
1. $(x+4)^2 = -8(y-3) \Rightarrow$ vertex of the parabola = $(-4,3)$; the parabola "opens" down and $p=-2 \Rightarrow$ focus = $(-4,3-2)=(-4,1)$; the center of the circle is the midpoint of the vertex and focus = $(-4,2)$.

2. $9x^2 - y^2 + 54x - 20y - 55 = 0 \Rightarrow 9\left(x^2+6x+9\right)-\left(y^2+20y+100\right)=55+81-100 \Rightarrow$

   $9(x+3)^2-(y+10)^2=36 \Rightarrow \dfrac{(x+3)^2}{4}-\dfrac{(y+10)^2}{36}=1$; C is a hyperbola in which $a^2=4$ and $b^2=36 \Rightarrow c^2=40 \Rightarrow c=2\sqrt{10} \Rightarrow 2c=4\sqrt{10}$.

3. $10x^2+y^2-30=0 \Rightarrow \dfrac{x^2}{3}+\dfrac{y^2}{30}=1 \Rightarrow E$ is an ellipse centered at the origin with the closest points being the co-vertices $\sqrt{3}$ units from the origin. Since P is $2\sqrt{3}$ units from the origin, its coordinates (x,y) satisfies the equation $x^2+y^2=12$. Subtracting this equation from $10x^2+y^2=30 \Rightarrow 9x^2=18 \Rightarrow x^2=2 \Rightarrow y^2=10 \Rightarrow (m,n)=\left(\sqrt{2},\sqrt{10}\right)$.

TEAM ROUND



1. $\dfrac{4}{3}p\,r^3=288p \Rightarrow r^3=216 \Rightarrow r=6$; examining the cross-sections of cone C with spheres S and T, presents the figure on the right; all the right triangles are 3-4-5 $\Rightarrow$ $\dfrac{6}{x}=\dfrac{3}{5} \Rightarrow x=10 \Rightarrow \dfrac{r}{4-r}=\dfrac{3}{5} \Rightarrow 5r=12-3r \Rightarrow r=\dfrac{3}{2} \Rightarrow$ $A=4p\left(\dfrac{3}{2}\right)^2=9p$.

2. Let p = probability Bill wins 1 game $\Rightarrow 2\,p$ = probability Al wins 1 game $\Rightarrow 6\,p$ = probability Carol wins 1 game; $p+2\,p+6\,p=1 \Rightarrow p=\dfrac{1}{9} \Rightarrow$ Al's probability = $\dfrac{2}{9}$ and Carol's = $\dfrac{2}{3}$; there are $\dfrac{4!}{2!\cdot 2!}=6$ ways Carol and Al can end up winning 2 games each $\Rightarrow$ the probability of this event happening = $6\left(\dfrac{2}{3}\right)^2\left(\dfrac{2}{9}\right)^2=\dfrac{32}{243}$.

3. There are 4×3 = 12 ways the first and last letters are consonants; consider what the 2nd, 3rd, and 4 letters could be: (i) EEE (1 permutation) (ii) EE and the A or one of the remaining 2 consonants (${}_3C_1\times\dfrac{3!}{2!}=9$ permutations) (iii) 1 or 0 E's $\Rightarrow$ 3 different letters (${}_4C_3\times 3!=24$ permutations); total permutations = $12(1+9+24)=408$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2004

ROUND 1 – Volume and Surface Area of Solids
Problem submitted by Maimonides

1. ____________________

2. ____________________

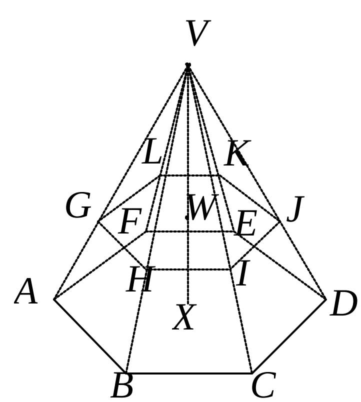
3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Two solid medal spheres of radius 4cm and 6cm are melted into a single sphere. Find the exact number of centimeters in the radius of this sphere.

2. Three cubes of volume 8 cm^3, 27 cm^3, and 125 cm^3 are glued together into a single solid of volume 160 cm^3. What is the minimum possible number of square centimeters in the surface area of this solid?

3. Regular hexagonal pyramid V-$ABCDEF$ on the right has each lateral edge of length $3\sqrt{10}$ cm and a height $\overline{VX}$ of length 9cm. Plane $GHIJKL$ is perpendicular to $\overline{VX}$ at point W and $VW:WX = 2:1$. Find the number of cubic centimeters in the volume of the frustum with vertices $ABCDEFGHIJKL$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2004

ROUND 2 – Inequalities and Absolute Value

Problem submitted by Belmont

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given $x^2 - 5x + 6 < 0$ and $y = x^2 + 5x + 6$, find all possible real values for *y*.

2. Find all real values of *x* satisfying the inequality, $|3x - 7| < 11 - 2x$.

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3. Find all real values of *x* satisfying the inequality, $\dfrac{\sqrt{4x^2 + 12x + 9}}{40 - x - 6x^2} > 0$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2004

ROUND 3 – Similar Polygons, Circles and Areas Related to Circles

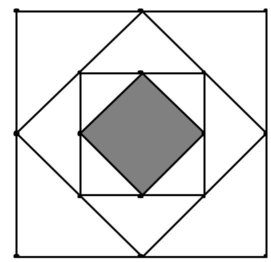
1. ____________________

2. ____________________

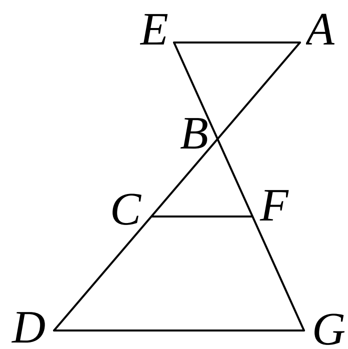
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

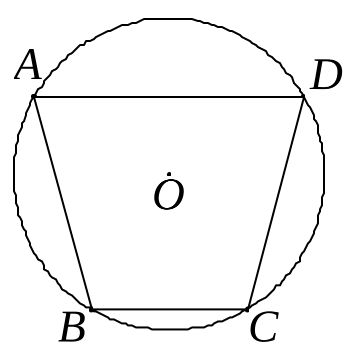
1. In the diagram on the right, the midpoints of the sides of one square are the vertices of another. If the sum of the areas of the four squares equals 120, find the perimeter of the shaded square.



2. Given parallel segments $\overline{EA}$, $\overline{CF}$, and $\overline{DG}$, $AB : BC : CD = 5 : 4 : 6$ and the difference between the areas of trapezoid $CFGD$ and ΔABE is 236, find the area of ΔBCF.



3. In circle O, $m\overset{\frown}{AB} = 90°$, $m\overset{\frown}{BC} = 60°$, $m\overset{\frown}{CD} = 90°$, and the area of quadrilateral $ABCD$ = 1, find the area of circle O.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2004

ROUND 4 – Sequences and Complex Numbers

Problem submitted by Belmont.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

Note in this round, $i = \sqrt{-1}$.

1. Solve the following equation over the set of complex numbers: $\left(x^2 + 2\right)^2 + x^2 = 0$

2. In an arithmetic sequence, its first term is 2, another term is 29, and the sum of all the terms from 2 to 29, inclusive, is 155. Find the common difference between terms.

3. In a geometric sequence $r = 1 + i$ and the sum of its first 12 terms is $\frac{65}{16}$, find its seventeenth term.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2004

ROUND 5 – Conics

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the conic *C*, whose equation is $4x^2 - 9y^2 + 36 = 0$, find the area of the circle which contains a diameter whose endpoints are the vertices of C.

2. Circle *O* with center $(2, -3)$ is externally tangent to circle P whose equation is $x^2 + y^2 + 8x - 6y + 17 = 0$. Find the radius of circle O.

3. Parabola *P* with focus $F(4, -8)$ contains points $C(6, -8)$, $A(a, 0)$ and $B(b, 0)$. Find the values for *a* and *b*.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2004

TEAM ROUND: Time limit: 12 minutes

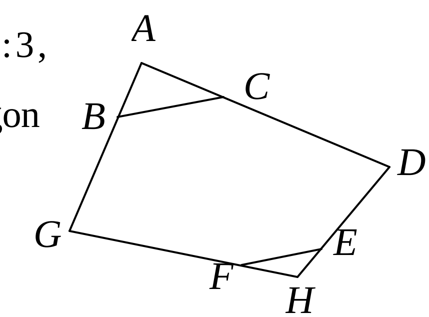
3 pts. 1. ________________

3 pts. 2. ________________

4 pts. 3. ________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given $AB:BG = AC:CD = 1:2$, $HF:FG = HE:ED = 1:3$, area of ΔABC = area of ΔFEH + 6, and the area of hexagon $BCDEFG$ = 209, find the area of quadrilateral $ADHG$.



2. Andy, Bob, and Carl are about to take their math final. Andy's probability of passing is $\frac{3}{4}$, Bob's probability of passing is $\frac{2}{3}$, and Carl's probability of failing is $\frac{1}{6}$. What is the probability that at least two of them pass the math final?

3. There are 6 urns, one of which contains 2 red and 4 blue marbles, two of which each contain 3 red and 3 blue marbles and the remaining urns each contain 4 red and 2 blue marbles. An urn is chosen at random and two marbles are randomly picked without replacement. What is the probability that both marbles are blue?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2004

ANSWER SHEET:

ROUND 1

1. $2\sqrt[3]{35}$ $\left(2\sqrt[3]{35}\text{cm}\right)$
2. 194 (194cm^2)
3. $\frac{57\sqrt{3}}{2}$ $\left(\frac{57\sqrt{3}}{2}\text{cm}^3\right)$

ROUND 2

1. $20 < y < 30$
2. $-4 < x < \frac{18}{5}$ or equivalent
3. $-\frac{8}{3} < x < \frac{5}{2}$ and $x \neq -\frac{3}{2}$ $\left(-\frac{8}{3} < x < -\frac{3}{2} \text{ or } -\frac{3}{2} < x < \frac{5}{2}\right)$

ROUND 3

1. $8\sqrt{2}$
2. 64
3. $\left(4-2\sqrt{3}\right)p$

ROUND 4

1. $\pm i, \pm 2i$
2. 3
3. $-16i$

ROUND 5

1. $4p$
2. $4\sqrt{2}$
3. $-2, 10$

TEAM ROUND

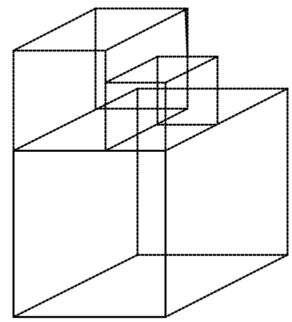
3 pts. 1. 229

3 pts. 2. $\frac{61}{72}$

4 pts. 3. $\frac{1}{6}$

ROUND 1 – Volumes and Surface Areas of Solids

1. The volumes of the two spheres are $\frac{4}{3}p \cdot 4^3$ and $\frac{4}{3}p \cdot 6^3$. When melted together the combined volume = $\frac{4}{3}p \cdot \left(4^3+6^3\right)$. Therefore the radius of this sphere = $\sqrt[3]{4^3+6^3} = 2\sqrt[3]{2^3+3^3} = 2\sqrt[3]{35}$.



2. In order to minimize its surface area the two smaller cubes with sides of length 2 cm and 3 cm are glued adjacent to each other on the same face of the largest cube with side of length 5 cm. The surface area of this solid = $6\left(5^2+3^2+2^2\right)-2\cdot 3^2-4\cdot 2^2=194$.

3. The distance from point X to a vertex of hexagon $ABCDEF = \sqrt{\left(3\sqrt{10}\right)^2-9^2}=3$ = side of hexagon $\Rightarrow$ area of hexagon = $\frac{3}{2}\left(3^2\sqrt{3}\right)=\frac{27\sqrt{3}}{2}$ $\Rightarrow$ volume of the pyramid = $\frac{1}{3}\left(\frac{27\sqrt{3}}{2}\right)(9)=\frac{81\sqrt{3}}{2}$; pyramid *V-GHIJKL* is similar to pyramid *V-ABCDEF* with the ratio of their sides equaling $\frac{2}{3}$ $\Rightarrow$ ratio of their volumes = $\left(\frac{2}{3}\right)^3=\frac{8}{27}$ $\Rightarrow$ volume of the frustum = $\frac{19}{27}\left(\frac{81\sqrt{3}}{2}\right)=\frac{57\sqrt{3}}{2}$.

ROUND 2 – Inequalities and Absolute Value

1. $x^2-5x+6<0 \Rightarrow (x-2)(x-3)<0 \Rightarrow 2<x<3$; the function $y=x^2+5x+6$ is increasing on this interval and since $y(2)=20$ and $y(3)=30$ $\Rightarrow 20<y<30$.

2. $|3x-7|<11-2x \Rightarrow$ if $x\geq\frac{7}{3}$, then $3x-7<11-2x \Rightarrow 5x<18 \Rightarrow x<\frac{18}{5}$; if $x\leq\frac{7}{3}$, then $7-3x<11-2x \Rightarrow -x<4 \Rightarrow x>-4$; the union of these intervals is $-4<x<\frac{18}{5}$.

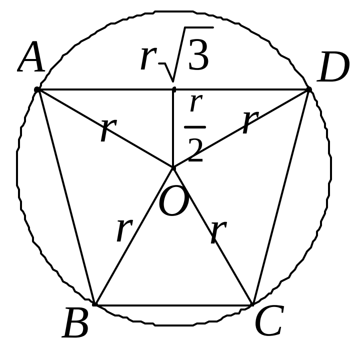
3. $\frac{\sqrt{4x^2+12x+9}}{40-x-6x^2}>0 \Rightarrow \frac{\sqrt{(2x+3)^2}}{6x^2+x-40}<0 \Rightarrow \frac{|2x+3|}{(3x+8)(2x-5)}<0 \Rightarrow x\neq -\frac{3}{2}$ and x is between the key numbers $-\frac{8}{3}$ and $\frac{5}{2}$ $\Rightarrow -\frac{8}{3}<x<\frac{5}{2}$ and $x\neq -\frac{3}{2}$.

ROUND 3 – Similar Polygons, Circles and Areas Related to Circles

1. Each square is half the area of the next larger. If A = area of the smallest square $\Rightarrow$ $A+2A+4A+8A=120 \Rightarrow 15A=120 \Rightarrow A=8 \Rightarrow$ side of the smallest square = $\sqrt{8}=2\sqrt{2}$ $\Rightarrow$ its perimeter = $8\sqrt{2}$.

2. Because of similar triangles, the ratio of the areas of $\Delta ABE : \Delta BCF : \Delta BDG = 25:16:100$. Let area of ΔABE = $25x$ $\Rightarrow$ area of trapezoid $CFGD$ = $100x-16x=84x$; $84x-25x=236 \Rightarrow 59x=236 \Rightarrow x=4 \Rightarrow$ area of $\Delta BCF = 16(4)=64$.

3. Let r = radius of the circle; area of ΔAOB = area of ΔCOD = $\frac{1}{2}r^2$; area of ΔBOC = area of ΔAOD = $\frac{\sqrt{3}}{4}r^2$;
$r^2\left(1+\frac{\sqrt{3}}{2}\right)=1 \Rightarrow r^2=\frac{2}{2+\sqrt{3}}=4-2\sqrt{3} \Rightarrow$
area of circle = $\left(4-2\sqrt{3}\right)p$.



ROUND 4 – Sequences and Complex Numbers

1. $\left(x^2+2\right)^2+x^2=0 \Rightarrow x^4+4x^2+4+x^2=0 \Rightarrow x^4+5x^2+4=0 \Rightarrow \left(x^2+4\right)\left(x^2+1\right)=0$
$\Rightarrow x^2=-1 \text{ or } -4 \Rightarrow x=\pm i, \pm 2i$.

2. Let n = number of terms $\Rightarrow \frac{n}{2}(2+29)=155 \Rightarrow n=\frac{310}{31}=10$;
$29=2+9d \Rightarrow 9d=27 \Rightarrow d=3$.

3. Let a_1 = first term: $a_1\left(\frac{1-(1+i)^{12}}{1-(1+i)}\right)=\frac{65}{16} \Rightarrow a_1\left(\frac{1-(2i)^6}{-i}\right)=\frac{65}{16} \Rightarrow a_1\left(\frac{1-(-64)}{-i}\right)=\frac{65}{16}$
$\Rightarrow a_1\left(\frac{65}{-i}\right)=\frac{65}{16} \Rightarrow a_1=\left(\frac{65}{16}\right)\left(\frac{-i}{65}\right)=\frac{-i}{16}$; $a_{17}=\left(\frac{-i}{16}\right)(1+i)^{16}=\left(\frac{-i}{16}\right)(2i)^8=\frac{-i}{2^4}\left(2^8\right)=-16i$.

ROUND 5 – Conics

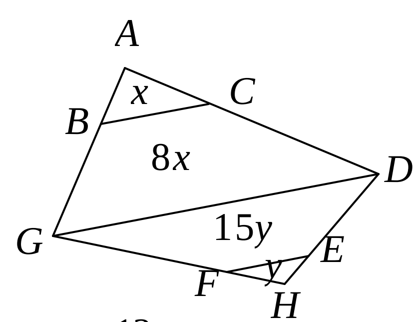
1. $4x^2 - 9y^2 + 36 = 0 \Rightarrow 9y^2 - 4x^2 = 36 \Rightarrow \dfrac{y^2}{4} - \dfrac{x^2}{9} = 1 \Rightarrow$ vertices of the hyperbola = $(0, \pm 2) \Rightarrow$ diameter = 4, the radius = 2, and the area of circle = $4p$.

2. $x^2 + y^2 + 8x - 6y + 17 = 0 \Rightarrow x^2 + 8x + 16 + y^2 - 6y + 9 = 8 \Rightarrow (x+4)^2 + (y-3)^2 = \left(2\sqrt{2}\right)^2$ $\Rightarrow$ its center = $(-4,3)$ and its radius = $2\sqrt{2}$; the radius of circle O = $\sqrt{(2+4)^2 + (-3-3)^2} - 2\sqrt{2} = 6\sqrt{2} - 2\sqrt{2} = 4\sqrt{2}$.

3. Since P has focus $F(4,-8)$, contains point $C(6,-8)$ and it has two x intercepts $\Rightarrow$ P opens up and since $FC = 2 \Rightarrow p = 1 \Rightarrow$ the vertex of the parabola = $(4,-9) \Rightarrow$ the equation of P is $(x-4)^2 = 4(1)(y+9)$. Set $y = 0$: $(x+4)^2 = 36 \Rightarrow$ $x + 4 = \pm 6 \Rightarrow x = -4 \pm 6 = -10, 2$.

TEAM ROUND

1. Draw $\overline{DG}$. Since $\Delta ABC \sim \Delta ADG$, the ratio of their areas = 1:9 and Since $\Delta EFH \sim \Delta DHG$, the ratio of their areas = 1:16. Let area of $\Delta ABC = x \Rightarrow$ area of trapezoid $BCDG = 8x$. Let area of $\Delta EFH = y \Rightarrow$ area of trapezoid $DEFG = 15y$. $x = y + 6$ and $8x + 15y = 209 \Rightarrow$ $8(y+6) + 15y = 209 \Rightarrow 23y + 48 = 209 \Rightarrow 23y = 161 \Rightarrow y = 7 \Rightarrow x = 13$. The area of quadrilateral = $9x + 16y = 9(13) + 16(7) = 229$.



2. The probability that at least two pass =
$$\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{5}{6}\right) + \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)\left(\frac{5}{6}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{3}\right)\left(\frac{5}{6}\right) = \frac{61}{72}$$

3. The probability of picking two blue marbles =
$$\frac{1}{6} \cdot \frac{{}_4C_2}{{}_6C_2} + \frac{2}{6} \cdot \frac{{}_3C_2}{{}_6C_2} + \frac{3}{6} \cdot \frac{{}_2C_2}{{}_6C_2} = \frac{6+6+3}{90} = \frac{1}{6}$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 1999

ROUND 1 – Arithmetic

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The retail price of an item underwent the following changes from year to year over a four year period: a 25% increase, a 25% decrease, a 20% increase, and finally a 20% decrease. **What percent increase or decrease** was the final retail price over the original retail price of the item? You must write the word **increase** or **decrease** as part of your answer.

2. Thirty children were polled about their likes or dislikes of Dawson's Creek or The Rugrats. The number of children who like Dawson's Creek is twice the number who like The Rugrats. The number who dislike Dawson's Creek equals the number who like The Rugrats. There are four children who like neither. Find the number of children who like Dawson's Creek and do not like The Rugrats.

3. Given the following base 5 addition:

$$\begin{array}{r} a\ \ b\ \ c\ \ d_{(5)} \\ +\ \ b\ \ d\ \ c_{(5)} \\ \hline a\ \ a\ \ d\ \ b_{(5)} \end{array}$$

where $a \neq 0$, $b \neq 0$ and a, b, c, and d are different digits, find the four digit number $a\,b\,c\,d_{(5)}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 1999

ROUND 2 – Algebra 1

1. $d =$ ____________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A club with x members raises a total of d dollars to spend equally at an amusement park. When one member cannot go to the amusement park, each member has an extra dollar to spend. Find d in terms of x.

2. Write the following expression in simplest radical form with the smallest possible index for the radical:

$$\frac{\left(\sqrt[3]{12}\right)\left(\sqrt[6]{72}\right)}{\left(\sqrt[4]{\sqrt[3]{108}}\right)\left(\sqrt{\sqrt[3]{3}}\right)}$$

3. Kaitlin is now eight years younger than half her mother's age. In k years (k a positive integer), Kaitlin will be one-third her mother's age then. What is the oldest possible age now for Kaitlin's mother?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 1999

ROUND 3 – Geometry

1. ____________________

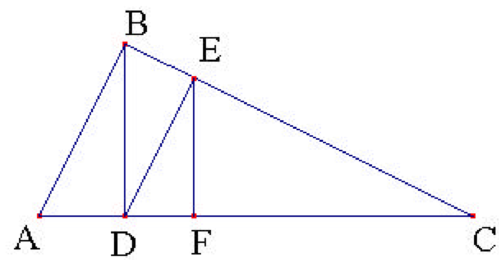
2. ____________________

3. ____________________

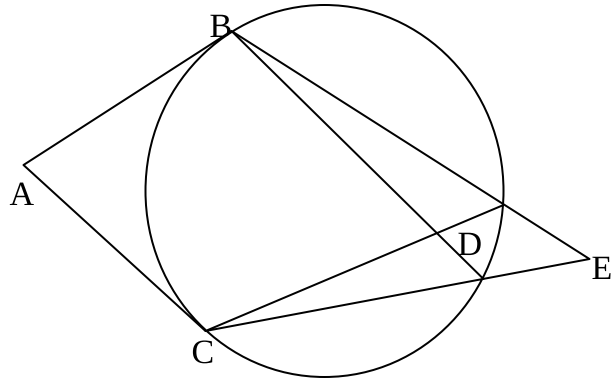
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. A square and an equilateral triangle have the same perimeter. If the triangle has an altitude of 6 units, how many units long is a diagonal of the square?

2. Given AD = 3, DC = 12, $\angle$ ABC, $\angle$ ADB, $\angle$ BED, and $\angle$ EFC are right angles, find the length of $\overline{EF}$.



3. Given $\overline{AB}$ and $\overline{AC}$ are tangent to the circle, m $\angle$ E = 42°, and m $\angle$ BDC = 66°, find the measure of $\angle$ A in degrees.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 1999

ROUND 4 – Algebra 2

1. $b=$ ______________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the quadratic equation in x, $x^2 - ax + b = 0$, such that the difference of its roots is 1, find b in terms of a.

2. The solution for x for the equation, $2^{2x-3} = 3^{2-x}$, can be put in the form $log_b a$ where a and b are positive integers. Under these conditions, find the smallest possible value for $a + b$.

3. Solve the following equation for x: $\sqrt[3]{8x + 16} + \sqrt[3]{x^2 + 4x + 4} = 15$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 1999

ROUND 5 – Precalculus

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the length of the line segment joining the center of circle C, which is $\left\{(x, y) \mid 2x^2 + 2y^2 - 4x + 12y + 4 = 0\right\}$ to the vertex of the parabola P, which is $\left\{(x, y) \mid y^2 = 8x + 4y + 12\right\}$

2. If $\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{4}$, find the value for $\sin(2x)$.

3. Some of the solutions to $z^{12} = -16$, when plotted in the complex plane, are located in quadrant I. Find the product of these solutions and write the result in the form $a + bi$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 1999

TEAM ROUND

3 pts. 1. ___________________

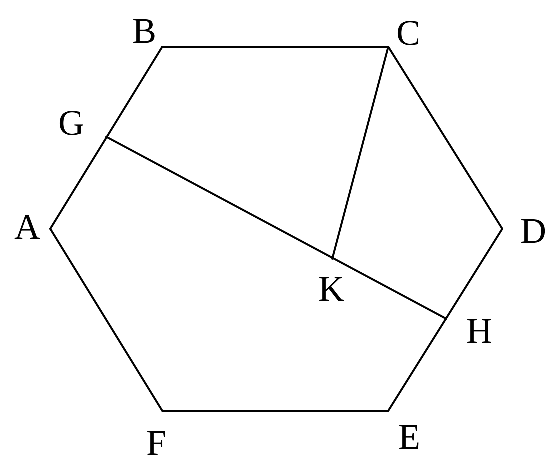
3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. Over the complex numbers, the solutions for x for the cubic equation $x^3 + 6x^2 + 21x + c = 0$ form an arithmetic sequence. Find the value for c.

2. Given regular hexagon, ABCDEF, 12 units on a side, with points G and H, midpoints of sides $\overline{AB}$ and $\overline{DE}$, and GK:KH = 2:1, find the area of quadrilateral BCKG. Write the answer in simplest radical form.

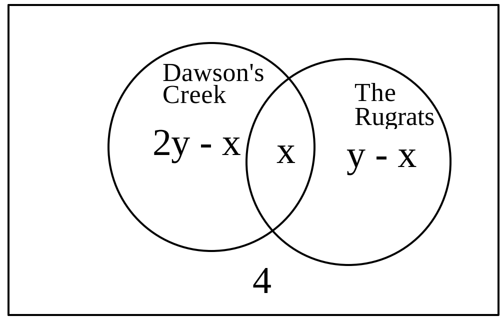


3. Urn A contains 6 red and 4 green marbles. Urn B contains 7 red and 3 green marbles. If 2 marbles are drawn at random from each urn, what is the probability that from those drawn, there will be 2 red and 2 green marbles? Write the answer in the form $\frac{a}{b}$ where a and b are relatively prime whole numbers.

Detailed Solutions of GBML for MEET 5 – MARCH 1999

ROUND 1

1. 25% increase, then a 25% decrease, then a 20% increase, and a 20% decrease $\Rightarrow$
$\frac{5}{4}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{4}{5}=\frac{9}{10}=$ **10% *decrease***



2. Liking The Rugrats = y; Liking Dawson's Creek = $2y$;
Liking both = x; $y - x + 4 = y \Rightarrow x = 4$;
$3y - 4 = 26 \Rightarrow y = 10$; Liking Dawson's Creek and not The Rugrats = $2y - x =$ **16**

3. $$\begin{array}{rcccl} & a & b & c & d_{(5)} \\ + & & b & d & c_{(5)} \\ \hline & a & a & d & b_{(5)} \end{array}$$
Since the last 2 columns are the same, but the sums are different $\Rightarrow$
$d + c > 4$ and since the 3rd column adds to $d \Rightarrow c = 4$; there is no carry from the 2nd column to the 1st column $\Rightarrow b = 1$ or 2; If $b = 2$, then the 2nd column would add to 0 because of the carry from the 3rd column $\Rightarrow b = 1 \Rightarrow a = 3 \Rightarrow d = 2$ since $d + 4 > 4$
$a\,b\,c\,d_{(5)} =$ **$3142_{(5)}$**

ROUND 2

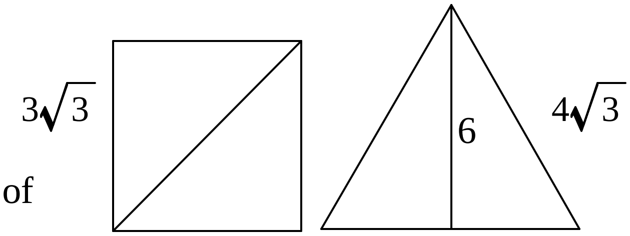
1. $\frac{d}{x-1}=\frac{d}{x}+1 \Rightarrow dx = dx - d + x(x-1) \Rightarrow d = \boldsymbol{x(x-1)}$ or $\boldsymbol{x^2 - x}$

2. $$\frac{\left(\sqrt[3]{12}\right)\left(\sqrt[6]{72}\right)}{\left(\sqrt[4]{\sqrt[3]{108}}\right)\left(\sqrt{\sqrt[3]{3}}\right)}=\frac{\left(2^2\cdot 3\right)^{1/3}\left(2^3\cdot 3^2\right)^{1/6}}{\left(2^2\cdot 3^3\right)^{1/12}\left(3\right)^{1/6}}=\frac{\left(2^{2/3}\cdot 3^{1/3}\right)\left(2^{1/2}\cdot 3^{1/3}\right)}{\left(2^{1/6}\cdot 3^{1/4}\right)\left(3\right)^{1/6}}=2^{2/3+1/2-1/6}\cdot 3^{2/3-1/4-1/6}=$$
$2\cdot 3^{1/4} = \boldsymbol{2\sqrt[4]{3}}$

3. Kaitlin's mother's age now = x; Kaitlin's age now = $0.5x - 8$;
Kaitlin's mother's age in k years = $x + k$; Kaitlin's age in k years = $0.5x - 8 + k$;
$3(0.5x - 8 + k) = x + k \Rightarrow 3x - 48 + 6k = 2x + 2k \Rightarrow x = 48 - 4k \Rightarrow x =$ **44** ($k = 1$)
is the oldest Kaitlin's mother can be.

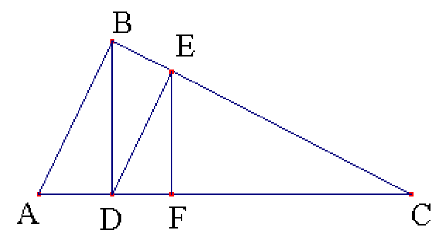
ROUND 3

1. side of equilateral triangle = $\frac{12}{\sqrt{3}} = 4\sqrt{3}$

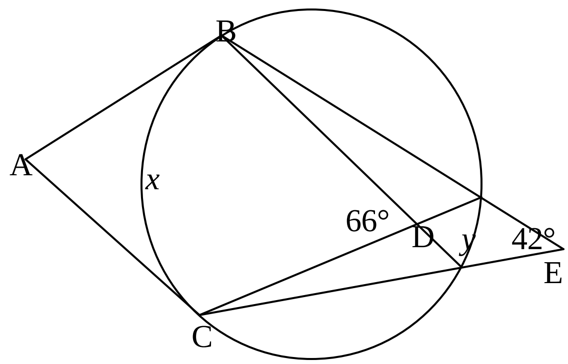


Perimeter of triangle = $12\sqrt{3}$ = Perimeter of of square $\Rightarrow$ side of square = $3\sqrt{3}$ $\Rightarrow$ diagonal of square = $3\sqrt{3}\sqrt{2} = \mathbf{3\sqrt{6}}$

2. $BD^2 = AD \times DC \Rightarrow BD = 6 \Rightarrow AB = 3\sqrt{5} \Rightarrow$ $6 : 3\sqrt{5} = DE : 6 \Rightarrow DE = 2.4\sqrt{5} \Rightarrow$ $EF : 2.4\sqrt{5} = 6 : 3\sqrt{5} \Rightarrow EF = \mathbf{4.8}$



3. $\frac{x+y}{2} = 66; \frac{x-y}{2} = 42; \Rightarrow$ $x = 108$; m $\angle$ A = 180° – 108° = 7**2°**



ROUND 4

1. Since the difference of the roots is 1, call the roots r and $r + 1$; $2r + 1 = a$ and $r(r+1) = b$ $\Rightarrow r = \frac{a-1}{2}$ and $b = \left(\frac{a-1}{2}\right)\left(\frac{a-1}{2} + 1\right) = \left(\frac{a-1}{2}\right)\left(\frac{a+1}{2}\right) = \mathbf{\frac{a^2-1}{4}}$ or $\mathbf{\frac{1}{4}a^2 - \frac{1}{4}}$

2. $2^{2x-3} = 3^{2-x} \Rightarrow (2x-3)\log 2 = (2-x)\log 3 \Rightarrow 2x \log 2 - 3\log 2 = 2\log 3 - x\log 3 \Rightarrow$ $x(2\log 2 + \log 3) = 3\log 2 + 2\log 3 \Rightarrow x \log 12 = \log 72 \Rightarrow x = \log_{12} 72 \Rightarrow a + b = \mathbf{84}$

3. $\sqrt[3]{8x+16} + \sqrt[3]{x^2+4x+4} = 15 \Rightarrow \sqrt[3]{8(x+2)} + \sqrt[3]{(x+2)^2} = 15 \Rightarrow$ $(x+2)^{2/3} + 2(x+2)^{1/3} - 15 = 0 \Rightarrow \left((x+2)^{1/3} + 5\right)\left((x+2)^{1/3} - 3\right) = 0 \Rightarrow$ $(x+2)^{1/3} = -5$ or $3 \Rightarrow x + 2 = -125$ or $27 \Rightarrow x = -\mathbf{127}$ or $\mathbf{25}$

ROUND 5

1. $x^2 + y^2 - 2x + 6y - 2 = 0 \Rightarrow (x-1)^2 + (y+3)^2 = 12 \Rightarrow$ center $= (1, -3)$
$y^2 - 4y + 4 = 8x + 16 \Rightarrow (y-2)^2 = 8(x+2) \Rightarrow$ vertex $= (-2, 2) \Rightarrow$ distance $= \mathbf{\sqrt{34}}$

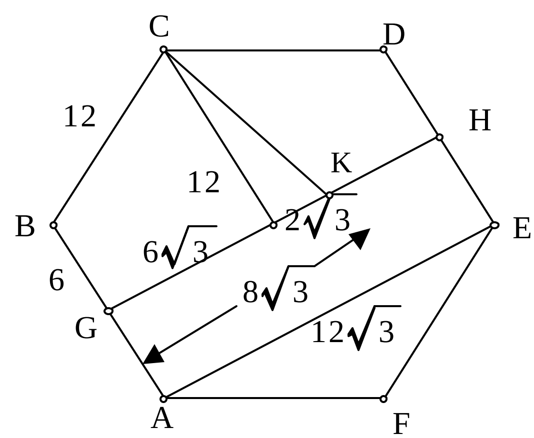
2. $sin\left(x + \frac{\pi}{4}\right) = \frac{1}{4} \Rightarrow sin\,x \cdot cos\,\frac{\pi}{4} + cos\;x \cdot sin\,\frac{\pi}{4} = \frac{1}{4} \Rightarrow sin\,x \cdot \frac{\sqrt{2}}{2} + cos\;x \cdot \frac{\sqrt{2}}{2} = \frac{1}{4} \Rightarrow$
$\frac{\sqrt{2}}{2}(sin\;x + cos\;x) = \frac{1}{4} \Rightarrow \frac{1}{2}(sin\;x + cos\;x)^2 = \frac{1}{16} \Rightarrow$
$\frac{1}{2}\left(sin^2\,x + 2sin\,x \cdot cos\;x + cos^2\;x\right) = \frac{1}{16} \Rightarrow \frac{1}{2}(1 + sin\,2x) = \frac{1}{16} \Rightarrow sin\,2x = -\mathbf{\frac{7}{8}}$

3. $z^{12} = -16 = 16\;cis\;180° \Rightarrow n = 0, 1, 2, \ldots 11 : z = 16^{\frac{1}{12}}cis\left(\frac{180°}{12} + 30°n\right) = 2^{\frac{1}{3}}cis\left(15° + 30°n\right)$
When $n = 0, 1,$ or 2: $z = 2^{\frac{1}{3}}cis\,15°, 2^{\frac{1}{3}}cis\,45°, 2^{\frac{1}{3}}cis\,75°$ Their product $= 2\;cis\;135° =$
$2\left(-\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = \mathbf{-\sqrt{2} + i\sqrt{2}}$

TEAM ROUND

1. Call the roots $r, r+d, r+2d$. The sum of the roots is the opposite of the coefficient for $x^2 \Rightarrow 3r + 3d = -6 \Rightarrow r + d = -2 \Rightarrow$ roots are $-2, -2+d$, and $-2-d$. The sum of the product of the roots taken 2 at a time is the coefficient of $x \Rightarrow$
$-2(-2+d) + -2(-2-d) + (-2+d)(-2-d) = 21 \Rightarrow 12 - d^2 = 21 \Rightarrow d = 3i \Rightarrow$ roots are
$-2, -2+3i, -2-3i$; c is the opposite of the product of the roots $\Rightarrow$
$c = 2(-2+3i)(-2-3i) = \mathbf{26}$

2. Draw a perpendicular from C to $\overline{GH}$ dividing the quadrilateral into a trapezoid and a right triangle.
GH = AE = $12\sqrt{3}$
The perpendicular is half the longest diagonal = 12
GK $= \frac{2}{3} \cdot 12\sqrt{3} = 8\sqrt{3}$
Area of triangle $= \frac{1}{2}\left(2\sqrt{3}\right)(12) = 12\sqrt{3}$; Area of trapezoid
$= \frac{1}{2}\left(6\sqrt{3}\right)(18) = 54\sqrt{3} \Rightarrow$ Area of quadrilateral $= \mathbf{66\sqrt{3}}$



3. There are 3 possibilities: (i) Urn A, 1 red and 1 green, and from Urn B, the same; (ii) Urn A 2 red, Urn B 2 green; (iii) Urn A 2 green, Urn B 2 red; $\Rightarrow$ Probability =

$$\frac{\binom{6}{1}\binom{4}{1}}{\binom{10}{2}} \times \frac{\binom{7}{1}\binom{3}{1}}{\binom{10}{2}} + \frac{\binom{6}{2}}{\binom{10}{2}} \times \frac{\binom{3}{2}}{\binom{10}{2}} + \frac{\binom{4}{2}}{\binom{10}{2}} \times \frac{\binom{7}{2}}{\binom{10}{2}} = \mathbf{\frac{1}{3}}$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 1999

ANSWER SHEET:

ROUND 1

1. 10% **decrease**
2. 16
3. $3142_{(5)}$ (3142 is acceptable)

ROUND 2

1. $x(x-1)$ *or* $x^2 - x$
2. $2\sqrt[4]{3}$
3. 44

ROUND 3

1. $3\sqrt{6}$
2. 4.8 or $\frac{24}{5}$ or $4\frac{4}{5}$
3. 72°

ROUND 4

1. $\frac{a^2-1}{4}$ or $\frac{1}{4}a^2 - \frac{1}{4}$
2. 84
3. –127, 25

ROUND 5

1. $\sqrt{34}$
2. $-\frac{7}{8}$
3. $-\sqrt{2} + i\sqrt{2}$

TEAM

3 pts. 1. 26

3 pts. 2. $66\sqrt{3}$

4 pts. 3. $\frac{1}{3}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2000

ROUND 1 – Arithmetic

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the 5-digit base ten number, $53{,}7T2$, is divisible by 12, find all possible values for *T*.

2. Given the following "*incomplete*" addition problem of 4-digit numbers in base *r*. Find the **sum** of all possible values for *r*.

$$
\begin{array}{cccc}
1 & 0 & 3 & 8_r \\
2 & 5 & 6 & 7_r \\
3 & 6 & 1 & 8_r \\
3 & 5 & 0 & 7_r \\
7 & 8 & 2 & 8_r \\
\hline
\ldots & \ldots & \ldots & 2_r
\end{array}
$$

3. How many 2-digit natural numbers have exactly 8 factors?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2000

ROUND 2 – Algebra 1

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Simplify the following expression:

$$\frac{x+1-\dfrac{2}{x+2}}{x-4-\dfrac{7}{x+2}}$$

2. Find the sum of all two-digit natural numbers whose tens' digit is three less than twice its units' digit.

3. Factor completely: $x^4 - x^3y - 2xy^3 - 4y^4$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2000

ROUND 3 – Geometry

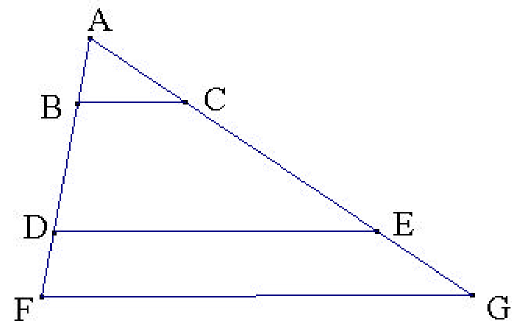
1. ____________________

2. ____________________

3. ____________________

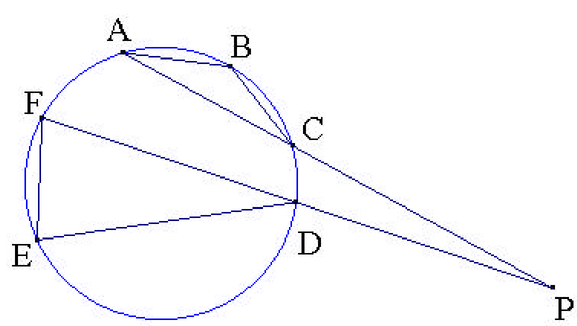
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given $\overline{BC}$ // $\overline{DE}$ // $\overline{FG}$, area (Δ ABC) = 4, area (BCED) = 32, and area (DEGF) = 28, find the ratio of DE to FG in simplified form.



2. Given a triangle all of whose sides are of integral lengths, with these three lengths equaling $2x$, $3x + 95$, and $6x + 19$, find how many distinct triangles can satisfy these conditions.

3. Given point A, B, C, D, E, and F on a circle such that m $\angle$ B = 135°, m $\angle$ E = 80°, and m $\angle$ P = 10°, find the ratio of $m\overset{\frown}{CD}$ to $m\overset{\frown}{AF}$ in simplified form.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2000

ROUND 4 – Algebra 2

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given a rectangle whose length is 8 cm longer than its width and the ratio of its area to its perimeter equals 3 cm^2: 2 cm , find the number of square centimeters in the area of this rectangle.

2. Solve the following equation for *x*. Put the result in simplest radical form.

$\log_3 2 + \log_9 7 = \log_{27} x$

3. Given the function, *f*, such that $f(x) = kx^2 + 6x + 4k$, find all real k such that the minimum value of *f* is positive.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2000

ROUND 5 – Precalculus

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the distance from F, the focus of the parabola, $\{(x, y) \mid y^2 - 8x + 16 = 0\}$ to circle C, $\{(x, y) \mid x^2 + y^2 + 12x + 11 = 0\}$.

2. Find the positive value for x satisfying the equation,

$$\cos(\text{Arctan } x) \cdot \tan\left(\text{Arccos}\left(\frac{2}{3}\right)\right) = \cos 660^\circ.$$

Note: Arctan and Arccos are names for inverse trigonometric functions.

3. The equation, $-2z^3 = (1 - i\sqrt{3})^4$ has complex solutions for z. Find **all** of these solutions in the polar form, $r \operatorname{cis} q$ where $0^\circ \leq q < 360^\circ$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2000

TEAM ROUND

3 pts. 1. __________________

3 pts. 2. __________________

4 pts. 3. __________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator or <u>any</u> calculator with symbolic operation capabilities**, which are not allowed on the Team Round

1. The lines $y = 2x + 6$, $y = mx$, and $y = 0$ intersect forming a triangle whose area equals 5. Solve for m.

2. Triangle ABC has vertices A (0, 0), B (6, 0) and C (18, 12). Find the distance PQ where P is the centroid and Q is the circumcenter of Δ ABC. If you estimate this distance, the result should be rounded to four decimal places.

3. Find the probability of drawing at random two cards from a standard deck containing no jokers such that at least one of the cards is a King and at least one of the cards is a diamond. If you estimate this probability, the result should be rounded to four decimal places.

Detailed Solutions of GBML for MEET 5 – MARCH 2000

ROUND 1

1. Since $53{,}7T2$ is divisible by both 3 and 4, 4 must divide evenly into the last 2 digits $\Rightarrow$ T could be 1, 3, 5, 7, or 9. The sum of the digits must be a multiple of 3 $\Rightarrow$ $17 + T$ is a multiple of 3 $\Rightarrow$ $T = 1$ or 7.

2. $r > 8$; the units' column adds to 38 $\Rightarrow$ $kr + 2 = 38$, for whole number k; $kr = 36$ $\Rightarrow$ r is a factor of 36 greater than 8 $\Rightarrow$ $r = 9, 12, 18, 36$ whose sum equals 75.

3. If a number has exactly 8 factors, it must be of the form: p^7, or p^3q, or pqr, where $p, q,$ and r are prime numbers. Since $2^7 > 99$, there are none of the first type; $2^3 \cdot 3,\ 2^3 \cdot 5,\ 2^3 \cdot 7,\ 2^3 \cdot 11,\ 3^3 \cdot 2$ are the only ones of the second type less than 100; $2 \cdot 3 \cdot 5,\ 2 \cdot 3 \cdot 7,\ 2 \cdot 3 \cdot 11,\ 2 \cdot 3 \cdot 13,\ 2 \cdot 5 \cdot 7$ are the only ones of the third type less than 100; therefore there are 10 possibilities altogether.

ROUND 2

1. $$\frac{x+1-\dfrac{2}{x+2}}{x-4-\dfrac{7}{x+2}} = \frac{(x+1)(x+2)-2}{(x-4)(x+2)-7} = \frac{x^2+3x}{x^2-2x-15} = \frac{x(x+3)}{(x+3)(x-5)} = \frac{x}{x-5}$$

2. $t = 2u - 3 \Rightarrow$ If $u = 2$, $t = 1$, if $u = 3$, $t = 3$, if $u = 4$, $t = 5$, if $u = 5$, $t = 7$, if $u = 6$, $t = 9$; $12 + 33 + 54 + 75 + 96 = 270$

3. $x^4 - x^3y - 2xy^3 - 4y^4 = x^4 - 4y^4 - x^3y - 2xy^3 =$
$\left(x^2 - 2y^2\right)\left(x^2 + 2y^2\right) - xy\left(x^2 + 2y^2\right) = \left(x^2 - xy - 2y^2\right)\left(x^2 + 2y^2\right) =$
$(x + y)(x - 2y)\left(x^2 + 2y^2\right)$

ROUND 3

1. area (Δ ADE) = 4 + 32 = 36; area (Δ AFG) = 4 + 32 + 28 = 64;
area (Δ ADE): area (Δ AFG) = 36: 64 ⇒ DE:FG = 6:8 = 3:4

2. Applying the triangle inequality theorem three times:
$2x+3x+95>6x+19 \Rightarrow x<76$; $2x+6x+19>3x+95 \Rightarrow 5x>76 \Rightarrow x>15.2$, and
$9x+114>2x \Rightarrow 7x>-114 \Rightarrow x>-16\tfrac{2}{7}$; the intersection of these three inequalities is:
$15.2<x<76 \Rightarrow x=16,17,...75$ which are 60 distinct cases.

3. Let $m\widehat{AF}=x$ and $m\widehat{CD}=y$; Since ∠ P is a *secant-secant* angle, $\dfrac{x-y}{2}=10 \Rightarrow x-y=20$
∠ B and ∠ E are inscribed ⇒ $m\widehat{AFC}=270°$ and $m\widehat{FAD}=160°$; the sum of these arcs include $\widehat{AF}$ and $\widehat{CD}$ twice $\Rightarrow 430=360+x+y \Rightarrow x+y=70$; Solving the two equations for *x* and *y* ⇒ $x=45$ and $y=25 \Rightarrow y:x=5:9$

ROUND 4

1. Call the width of the rectangle *x* ⇒ length is *x* + 8 ⇒ area is $x(x+8)$ and the perimeter is $4x+16$; $\dfrac{x^2+8x}{4x+16}=\dfrac{3}{2} \Rightarrow x^2+8x=6x+24 \Rightarrow x^2+2x-24=0 \Rightarrow$
$(x+6)(x-4)=0 \Rightarrow x=4 \Rightarrow area=4\cdot 12=48$

2. $\log_3 2+\log_9 7=\log_{27} x \Rightarrow \dfrac{\log 2}{\log 3}+\dfrac{\log 7}{\log 9}=\dfrac{\log x}{\log 27} \Rightarrow \dfrac{\log 2}{\log 3}+\dfrac{\log 7}{2\log 3}=\dfrac{\log x}{3\log 3} \Rightarrow$
$\left(\dfrac{\log 2}{\log 3}+\dfrac{\log 7}{2\log 3}\right)6\log 3=\left(\dfrac{\log x}{3\log 3}\right)6\log 3 \Rightarrow 6\log 2+3\log 7=2\log x \Rightarrow$
$\log\left(2^6\cdot 7^3\right)=\log x^2 \Rightarrow x=\sqrt{2^6\cdot 7^3}=2^3\cdot 7\sqrt{7}=56\sqrt{7}$ [Note *x* must be greater than 0.]

3. The minimum value for the quadratic function occurs at its vertex. ⇒ $x=-\dfrac{b}{2a}$
$\Rightarrow x=-\dfrac{3}{k} \Rightarrow f\left(-\dfrac{3}{k}\right)=k\left(-\dfrac{3}{k}\right)^2+6\left(-\dfrac{3}{k}\right)+4k=\dfrac{9}{k}-\dfrac{18}{k}+4k=4k-\dfrac{9}{k}$;
$4k-\dfrac{9}{k}>0 \Rightarrow \dfrac{4k^2-9}{k}>0 \Rightarrow \dfrac{(2k-3)(2k+3)}{k}>0$; key numbers for this inequality are
$-\dfrac{3}{2}$, 0, $\dfrac{3}{2}$; Since the parabola opens up, $k>0$, The values for *k* that makes the fraction greater than 0 in relation to these conditions are: $k>\dfrac{3}{2}$

ROUND 5

1. $y^2 - 8x + 16 = 0 \Rightarrow y^2 = 8(x-2) \Rightarrow \textit{vertex is } (2,0) \textit{ and } p = 2$; since the parabola opens to the right $\Rightarrow F = (2+2,0) = (4,0)$; $x^2 + y^2 + 12x + 11 = 0 \Rightarrow x^2 + 12x + 36 + y^2 = 25 \Rightarrow (x+6)^2 + y^2 = 5^2 \Rightarrow$ center of the circle is $(-6,0)$ and the radius is 5 $\Rightarrow$ the closest point on the circle to $(4,0)$ is $(-1,0)$ and the distance is 5.

2. Let $a = \text{Arctan}\, x \Rightarrow \tan a = x \Rightarrow \cos a = \dfrac{1}{\sqrt{1+x^2}}$; Let $b = \text{Arccos}\left(\dfrac{2}{3}\right) \Rightarrow \cos b = \dfrac{2}{3}$

$\Rightarrow \tan b = \dfrac{\sqrt{5}}{2}; \cos 660° = \cos 300° = \dfrac{1}{2}; \dfrac{1}{\sqrt{1+x^2}} \cdot \dfrac{\sqrt{5}}{2} = \dfrac{1}{2} \Rightarrow \sqrt{5} = \sqrt{1+x^2} \Rightarrow x = 2$

3. $-2 = 2\,cis 180°; 1 - i\sqrt{3} = 2\,cis\,300° \Rightarrow \left(1 - i\sqrt{3}\right)^4 = 16\,cis 1200° = 16\,cis\,120°;$

$z^3 = \dfrac{16\,cis 120°}{2\,cis 180°} = 8\,cis(-60°) = 8\,cis\,300° \Rightarrow z = 2\,cis 100°, 2\,cis\,220°, 2\,cis\,340°$

or $z = -2\,cis\,40°, -2\,cis 160°, -2\,cis\,280°$

TEAM ROUND

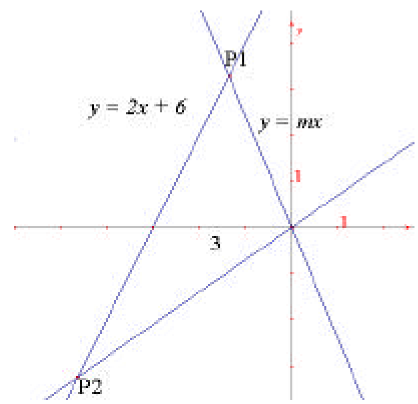
1. Find y coordinate of point P:

$2x + 6 = mx \Rightarrow x = \dfrac{6}{m-2} \Rightarrow$

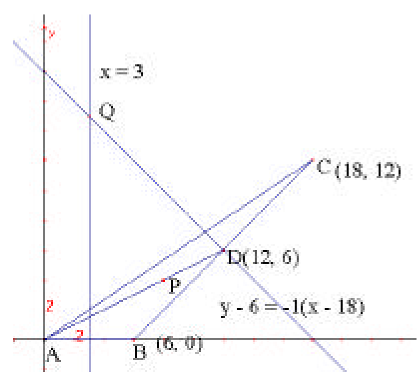
$y = \dfrac{6m}{m-2} \Rightarrow$ area of the triangle $= \dfrac{1}{2} \cdot 3 \cdot \dfrac{6m}{m-2} = \dfrac{9m}{m-2}$

$= \pm 5 \Rightarrow 9m = 5m - 10 \Rightarrow m = -\dfrac{5}{2}$ (using point P1)

or $9m = -5m + 10 \Rightarrow m = \dfrac{5}{7}$ (using point P2)



2. To find centroid P: 2/3 the distance from A to D (12, 6), the midpoint of BC. $\Rightarrow$
P has coordinates (8, 4); to find circumcenter Q : $x = 3$ is $\perp$ bis. of $\overline{AB}$; slope of $\overline{BC} = 1 \Rightarrow$
$y - 6 = -1(x - 12)$ is $\perp$ bis. of $\overline{BC}$; $\Rightarrow$
$y - 6 = -1(3 - 12) \Rightarrow y = 15 \Rightarrow$ Q has coordinates (3, 15); PQ = $\sqrt{5^2 + (-11)^2} = \sqrt{146} \approx 12.0830$



3. two cases: (i) draw a King, which is not a diamond and then a diamond (ii) draw the diamond King and then any card except a second King $\therefore$ the probability equals

$$\frac{\binom{3}{1}\binom{13}{1} + \binom{1}{1}\binom{48}{1}}{\binom{52}{2}} = \frac{29}{442} \approx 0.0656$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2000

ANSWER SHEET:

ROUND 1

1. 1, 7
2. 75
3. 10

ROUND 2

1. $\frac{x}{x-5}$
2. 270
3. $(x+y)(x-2y)(x^2+2y^2)$

ROUND 3

1. 3:4 $\left(or\ \frac{3}{4}\right)$
2. 60
3. 5:9 $\left(or\ \frac{5}{9}\right)$

ROUND 4

1. 48
2. $56\sqrt{7}$
3. $k > \frac{3}{2}$

ROUND 5

1. 5
2. 2
3. $2\,cis100°, 2\,cis220°, 2\,cis340°$

   $(or\ -2\,cis40°, -2\,cis160°, -2\,cis280°)$

TEAM

3 pts. 1. $-\frac{5}{2}, \frac{5}{7}$

3 pts. 2. $\sqrt{146}$ $(or\ 12.0830)$

4 pts. 3. $\frac{29}{442}$ $(or\ 0.0656)$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2001

ROUND 1 – Arithmetic
Problem submitted by Newton South.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. For what value of k is the following equation true?

$$2000^2 + 2000^3 = 2001k$$

2. Given the following multiplication in base 8, find the base 10 value of the number xy_8.

$$\begin{array}{r} xy_8 \\ \underline{2} \\ yx_8 \end{array}$$

3. The greatest common factor of two whole numbers is 12 and their least common multiple is 2520. Find the smallest possible value for their sum.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2001

ROUND 2 – Algebra 1

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Jill has three times more money than Jack. If Jill gives Jack $15, Jill will now have twice Jack's new amount. How many dollars did Jill start with?

2. Two numbers differ by 1 and the sum of their reciprocals equals $\frac{15}{4}$. Find all possible values for the smaller of these two numbers.

3. There are only two lines containing the point $P(9,-1)$ that form a triangle with the x and y axes with an area of 6 square units. Find the slopes of these two lines.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2001

ROUND 3 – Geometry

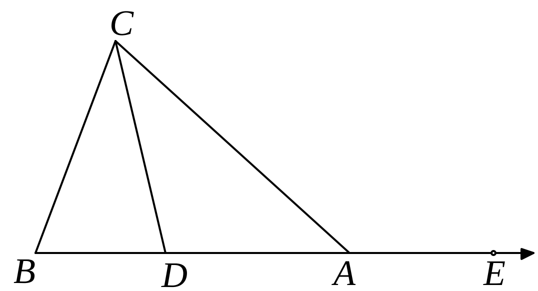
1. ____________________

2. ____________________

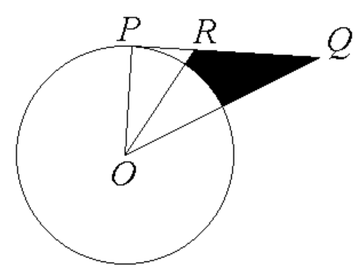
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

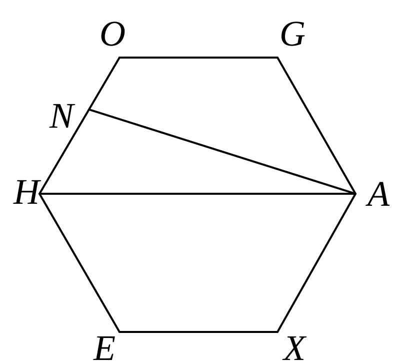
1. Given $AB = AC$, $\overline{BDAE}$, and $\overline{CD}$ bisects $\angle ACB$. If $m\angle CDE + m\angle CAE = 245°$, find the number of degrees in $m\angle B$.



2. Given circle O with radius of length 6 cm, $\overline{PRQ}$, $\overline{PQ}$ tangent to circle O at point P, $\overline{OR}$ bisects $\angle POQ$, and $OR = RQ$, find the number of square centimeters in the shaded area indicated on the diagram.



3. Given $HEXAGO$ is a regular hexagon, and $ON : NH = 3:5$. If the area of the regular hexagon is $24\sqrt{3}$ square inches, find the number of square inches in the area of quadrilateral $AGON$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2001

ROUND 4 – Algebra 2

1. ____________________

2. (,) ____________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all values for *x* satisfying the following equation: $\log_{2/3}(x+3) = -2 + \log_{2/3}(x-2)$

2. Find the ordered pair (x, y), where *x* and *y* are rational, satisfying the following equation: $12^{x+y} = 6 \cdot 18^{x-2y}$

3. Find all values for *x* satisfying the following inequality: $\dfrac{x}{|x-2|} > 2$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2001

ROUND 5 – Precalculus

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the conic C , $\{(x,y)\,|\,x^2+y^2-x-4y-12=0\}$, find the length of the chord whose endpoints are where the y axis intersects C.

2. Given $\cos x=\frac{1}{7}, \frac{3p}{2}<x<2p$, find the value for $\cos\left(x+\frac{2p}{3}\right)$.

3. Given $z=c+di,\ c>0,\ d>0$, and $z^4=2-2i\sqrt{3}$, find the value for $\frac{2z}{1+i}$ in $a+bi$ form.

   Note: $i=\sqrt{-1}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2001

TEAM ROUND

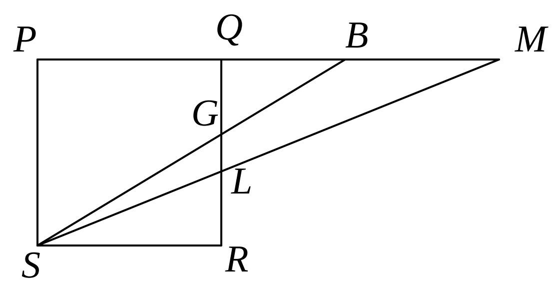
3 pts. 1. ___________________

3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND except for the **TI-89 Calculator or <u>any</u> calculator with symbolic operation capabilities**, which are not allowed on the Team Round

1. $PQRS$ is a square. $\overline{PQBM}$, $\overline{QGLR}$, $\overline{SGB}$, $\overline{SLM}$, $QG:GR = 2:3$, and $QL:LR = 3:2$. Find the ratio of the area of quadrilateral $GBML$ to the area of the square $PQRS$.



2. If the cubic equation $x^3 - 3x^2 + 2x + k = 0$, when solved over the complex numbers, has roots *r*, *s*, and *t*, and $(r+2)(s+2)(t+2) = 17$, find the value for *k*.

3. Three cards are picked at random from a standard deck of cards (no jokers). What is the probability that only one of them is a face card and only one of them is a heart? Express the result in the form $\frac{a}{b}$ where *a* and *b* are relative prime whole numbers or, if estimated, round off to 4 decimal places.

Detailed Solutions of GBML for MEET 5 – MARCH, 2001

ROUND 1

1. $2000^2+2000^3=2001k \rightarrow 2000^2(1+2000)=2001k \rightarrow k=2000^2=4000000$

2. $2(8x+y)=8y+x \rightarrow 16x+2y=8y+x \rightarrow 15x=6y \rightarrow 5x=2y \rightarrow$
$x=2,\ y=5 \rightarrow 8x+y=21$

3. Since their GCF = 12, the numbers are of the form 12x and 12y. Since their LCM =2520, $(12x)(12y)=12\cdot 2520 \rightarrow xy=210$. The smallest value for $x+y$ is when the factors of 210 have the smallest difference $\rightarrow$ $x=14,\ y=15 \rightarrow 12x+12y=12\cdot 29=348$.

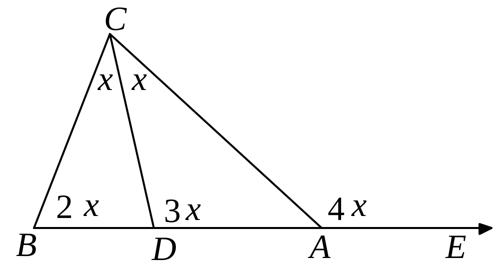
ROUND 2

1. Let x = Jack's original amount $\rightarrow$ 3x = Jill's $\rightarrow$
$3x-15=2(x+15) \rightarrow 3x-15=2x+30 \rightarrow x=45 \rightarrow 3x=135$

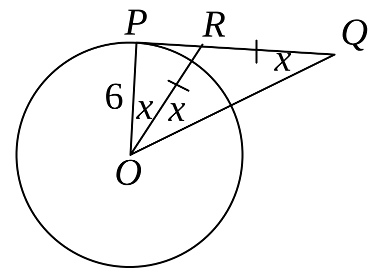
2. $\frac{1}{x}+\frac{1}{x+1}=\frac{15}{4} \rightarrow 4x+4+4x=15x^2+15x \rightarrow 15x^2+7x-4=0 \rightarrow$
$(5x+4)(3x-1)=0 \rightarrow x=-\frac{4}{5},\frac{1}{3}$

3. $y+1=m(x-9) \rightarrow$ if $x=0 \rightarrow y=-9m-1$ or if $y=0 \rightarrow x=\frac{1}{m}+9$;
the area of the triangle = $\frac{1}{2}(-9m-1)\left(\frac{1}{m}+9\right)=6 \rightarrow -9-81m-\frac{1}{m}-9=12 \rightarrow$
$81m+\frac{1}{m}+30=0 \rightarrow 81m^2+30m+1=0 \rightarrow (27m+1)(3m+1)=0 \rightarrow m=-\frac{1}{3},-\frac{1}{27}$

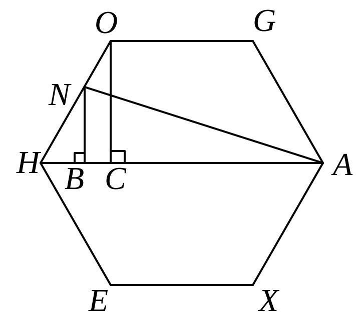
ROUND 3



1. Call $m\angle BCD = x \rightarrow m\angle ACD = x \rightarrow m\angle B = 2x$ $\rightarrow m\angle DCA = 3x$ and $m\angle CAE = 4x \rightarrow$ $3x + 4x = 245 \rightarrow 7x = 245 \rightarrow x = 35 \rightarrow 2x = 70$



2. $3x = 90 \rightarrow x = 30 \rightarrow PQ = 6\sqrt{3}$ and $PR = 2\sqrt{3} \rightarrow$ $RQ = 4\sqrt{3}$; shaded area = area of ΔORQ – area of $30°$ sector = $\frac{1}{2}\left(4\sqrt{3}\right)(6) - \frac{30}{360}p \cdot 6^2 = 12\sqrt{3} - 3p$.



3. Let s = side of hexagon: $\frac{3}{2}s^2\sqrt{3} = 24\sqrt{3} \rightarrow s^2 = 16 \rightarrow s = 4$.
Draw perpendiculars from O and N to $\overline{AH}$.
$s = 4 \rightarrow AH = 8,\ OC = 2\sqrt{3} \rightarrow NB = \frac{5}{8}\left(2\sqrt{3}\right) = \frac{5\sqrt{3}}{4}$.
Area of $AGON$ = area of $AGOH$ – area of ΔAHN
$= \frac{1}{2}\left(24\sqrt{3}\right) - \frac{1}{2}(8)\left(\frac{5\sqrt{3}}{4}\right) = 12\sqrt{3} - 5\sqrt{3} = 7\sqrt{3}$.

ROUND 4

1. $\log_{2/3}(x+3) = -2 + \log_{2/3}(x-2) \rightarrow 2 = \log_{2/3}(x-2) - \log_{2/3}(x+3) \rightarrow$
$2 = \log_{2/3}\left(\frac{x-2}{x+3}\right) \rightarrow \frac{x-2}{x+3} = \frac{4}{9} \rightarrow 9x - 18 = 4x + 12 \rightarrow 5x = 30 \rightarrow x = 6$.

2. $12^{x+y} = 6 \cdot 18^{x-2y} \rightarrow \left(2^2 \cdot 3\right)^{x+y} = (2 \cdot 3)\left(2 \cdot 3^2\right)^{x-2y} \rightarrow 2^{2x+2y} \cdot 3^{x+y} = 2^1 \cdot 3^1 \cdot 2^{x-2y} \cdot 3^{2x-4y}$
$\rightarrow \begin{cases} 2x + 2y = 1 + x - 2y \\ x + y = 1 + 2x - 4y \end{cases} \rightarrow \begin{cases} x + 4y = 1 \\ -x + 5y = 1 \end{cases} \rightarrow 9y = 2 \rightarrow y = \frac{2}{9} \rightarrow x = \frac{1}{9} \rightarrow (x, y) = \left(\frac{1}{9}, \frac{2}{9}\right)$.

3. $\frac{x}{|x-2|} > 2 \rightarrow x \neq 2$ and since $|x-2| > 0 \rightarrow x > 2|x-2|$;
if $x > 2$: $x > 2(x-2) \rightarrow x > 2x - 4 \rightarrow -x > -4 \rightarrow x < 4$;
if $x < 2$: $x > 2(2-x) \rightarrow x > 4 - 2x \rightarrow 3x > 4 \rightarrow x > \frac{4}{3}$; therefore the solution to the inequality is $\frac{4}{3} < x < 4$ and $x \neq 2$ or equivalently $\frac{4}{3} < x < 2$ or $2 < x < 4$

ROUND 5

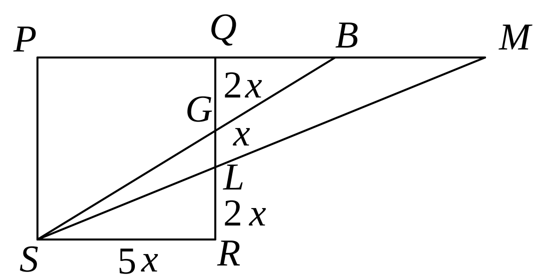
1. To find the endpoints of the chord on the circle set $x = 0$:
$y^2 - 4y - 12 = 0 \rightarrow (y-6)(y+2) = 0 \rightarrow y = -2, 6 \rightarrow$ length of chord $= 6 - (-2) = 8$.

2. $\cos x = \frac{1}{7}, \frac{3p}{2} < x < 2p \rightarrow \sin x = -\sqrt{1 - \left(\frac{1}{7}\right)^2} = -\sqrt{\frac{48}{49}} = -\frac{4\sqrt{3}}{7}$; $\cos\left(x + \frac{2p}{3}\right) =$
$\cos x \cdot \cos\left(\frac{2p}{3}\right) - \sin x \cdot \sin\left(\frac{2p}{3}\right) = \left(\frac{1}{7}\right)\left(-\frac{1}{2}\right) - \left(-\frac{4\sqrt{3}}{7}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{-1+12}{14} = \frac{11}{14}$.

3. $2 - 2i\sqrt{3} = 2\left(1 - i\sqrt{3}\right) = 2(2\text{cis}300°) = 4\text{cis}300°$;
since $z = c + di$, $c > 0$, $d > 0$, and $z^4 = 4\text{cis}300° \rightarrow z = 4^{1/4}\text{cis}75° = \sqrt{2}\text{cis}75°$;
$1 + i = \sqrt{2}\text{cis}45° \rightarrow \frac{2z}{1+i} = \frac{2\sqrt{2}\text{cis}75°}{\sqrt{2}\text{cis}45°} = 2\text{cis}30° = \sqrt{3} + i$

TEAM ROUND

1. Let side of square = $5x \rightarrow$ because of the ratios, $QG = 2x$, $GL = x$, $LR = 2x$; area of $GBML$ = area of ΔQML – area of ΔQBG; $\Delta QML \sim \Delta RSL$ with ratio of similitude 3:2; area of $\Delta\ RSL = 5x^2$ $\rightarrow$ area of $\Delta\ QML = \frac{9}{4} \cdot 5x^2 = \frac{45}{4}x^2$;



$\Delta QBG \sim \Delta RSG$ with ratio of similitude 2:3; area of $\Delta\ RSG = \frac{15x^2}{2} \rightarrow$ area of ΔQBG =
$\frac{4}{9} \cdot \frac{15}{2}x^2 = \frac{10}{3}x^2$; ratio of area of quad $GBML$: square = $\frac{\frac{45}{4} - \frac{10}{3}}{25} = \frac{19}{60}$

2. $x^3 - 3x^2 + 2x + k = 0 \rightarrow r + s + t = 3$, $rs + rt + st = 2$, and $rst = -k$;
$(r+2)(s+2)(t+2) = 17 \rightarrow rst + 2(rs + st + rt) + 4(r + s + t) + 8 = 17 \rightarrow$
$-k + 2(2) + 4(3) + 8 = 17 \rightarrow k = 7$

3. There are two types of successful events: (i) 1 card is a non-heart face card, 1 card is a non-face card heart, and 1 card is a non-heart, non-face card; (ii) 1 card is a heart and a face card and 2 cards are non-hearts, non-face cards; there are 9 non-heart face cards, 10 non-face card hearts, 30 non-hearts, non-face cards, and 3 hearts and face cards;
therefore the probability = $\frac{{}_9C_1 \cdot {}_{10}C_1 \cdot {}_{30}C_1 + {}_3C_1 \cdot {}_{30}C_2}{{}_{52}C_3} = \frac{801}{4420} \approx 0.1812$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2001

ANSWER SHEET:

ROUND 1

1. 4,000,000
2. 21
3. 348

ROUND 2

1. 135 (or \$135)
2. $-\frac{4}{5}, \frac{1}{3}$
3. $-\frac{1}{3}, -\frac{1}{27}$

ROUND 3

1. 70 (or 70°)
2. $12\sqrt{3} - 3p$ $\left(\text{or } 12\sqrt{3} - 3p \text{ cm}^2\right)$
3. $7\sqrt{3}$ $\left(\text{or } 7\sqrt{3}\text{in}^2\right)$

ROUND 4

1. 6
2. $\left(\frac{1}{9}, \frac{2}{9}\right)$
3. $\frac{4}{3} < x < 4$ and $x \neq 2$
   $\left(\frac{4}{3} < x < 2 \text{ or } 2 < x < 4\right)$

ROUND 5

1. 8
2. $\frac{11}{14}$
3. $\sqrt{3} + i$

TEAM

3 pts. 1. $19:60$ $\left(\text{or } \frac{19}{60}\right)$

3 pts. 2. 7

4 pts. 3. $\frac{801}{4420} \approx 0.1812$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2002

ROUND 1 – Arithmetic
Problems submitted by Newton South and Maimonides.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A fraction has the value $\frac{12}{13}$. If the sum of the numerator and denominator is 2000, find the fraction's denominator.

2. Find the only whole number that equals 12 times the sum of its digits.

3. Given N is a factor of 2002 and N has exactly 4 factors. How many possible values for N are there?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2002

ROUND 2 – Algebra 1
Problems submitted by Newton South.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the set of all $x, x \in \Re$,which are the solution for $\left\{ x \,\middle|\, \frac{x+2}{x-2} > 5 \right\}$.

2. Given points $A(6,-2)$, $B(t+1,-4)$, and $C(t,4)$ such that $\angle BAC$ is a right angle, find all possible values for *t*.

3. Ticket prices for the afternoon movie are \$5.50 for adults, \$4.50 for children and \$4.00 for senior citizens. If 100 tickets were sold, the proceeds were \$465, and more senior citizens than children attended, find the most number of children that could have been at the afternoon movie.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2002

ROUND 3 – Geometry
Problems submitted by Maimonides.

1. ____________________

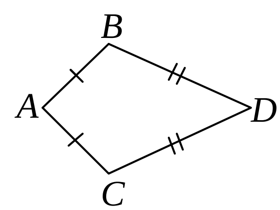
2. ____________________

3. ____________________

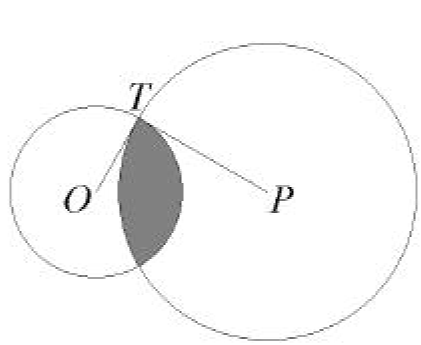
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. An isosceles triangle has a perimeter of 111 cm and all of its sides are of integral length. There are N of this type of triangles. Find N.

2. A kite (See the figure on the right with indicated equal sides.) has diagonals whose lengths are in the ratio of 5:2. The area of the kite is 4 square centimeters. If a circle can be circumscribed about this kite, find the number of square centimeters in the area of the circle.



3. Given circles O and P with radii of length $\sqrt{6}$ and $3\sqrt{2}$ inches respectively. If T is a point of intersection of the two circles, and $\overline{PT}$ is tangent to circle O, find the number of square inches in the shaded area, which is the area common to both circles.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2002

ROUND 4 – Algebra 2
Problems submitted by Maimonides.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Solve the following equation for *x*: $\log_{25} x = 18\log_{x^4} 5$.

2. Four positive numbers form a geometric sequence. The sum of these four numbers divided by the sum of first two numbers is 37. If the first number is a, find the fourth number in terms of a.

3. If k is added to each of the numbers 4, 124, and 316, the results are the squares of consecutive terms of an arithmetic sequence. Solve for k.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2002

ROUND 5 – Precalculus
Problems submitted by Maimonides.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the radian measure for x given that $x = \text{Arctan}\left(\frac{1}{2}\right) + \text{Arctan}\left(\frac{1}{3}\right)$. Note that Arctan is the inverse tangent function.

2. In ΔABC, $AB = 1$, $AC = 5$, and the area of ΔABC = 2, find all possible values for the length of $\overline{BC}$.

3. The angle with measure –30° is drawn in *standard position* in the coordinate plane. Find the point of intersection of the terminal side of this angle with the conic having vertices $\left(\pm\sqrt{6}, 0\right)$ and foci $(\pm 3, 0)$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2002

TEAM ROUND (12 MINUTES LONG)

Problems submitted by Maimonides.

3 pts. 1. ___________________

3 pts. 2. ___________________

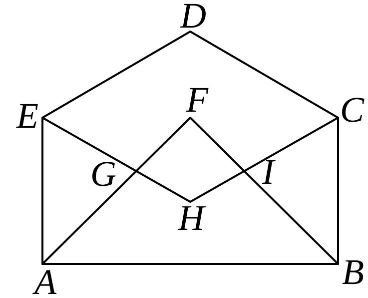
4 pts. 3. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND except for the TI-89 Calculator or any calculator with symbolic operation capabilities, which are not allowed on the Team Round

1. Kaitlin picked a whole number greater than 10, added one, multiplied this result by two, added one again, multiplied this result by three, added one for the last time, and multiplied this result by four. If Kaitlin ended with a perfect cube, find the smallest two whole numbers greater than 10 that she could have picked.

2. Al and Marty play a game where each of them tosses 4 fair coins and whoever has more coins landing on heads wins. What is the probability that Marty ties the first game and wins the second game? Express the answer in rational form or if estimated round off to exactly 5 decimal places.

3. Given pentagon $ABCDE$ with $m\angle BAE = m\angle ABC = 90°$, $m\angle AED = m\angle BCD = 120°$, $AB = 12$, $AE = BC = 6$, $\overline{AF}$, $\overline{BF}$, $\overline{CH}$, and $\overline{EH}$ bisect angles $BAE, ABC, BCD,$ and AED respectively, find the exact area of quadrilateral $FGHI$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2002

ANSWER SHEET:

ROUND 1

1. 1040
2. 108
3. 6

ROUND 2

1. $2 < x < 3$
2. 2, 9
3. 31

ROUND 3

1. 28
2. 5π (or 5π cm^2)
3. $5p - 6\sqrt{3}$ $\left(\text{or } 5p - 6\sqrt{3}\text{in}^2\right)$

ROUND 4

1. $125, \frac{1}{125}$ $\left(\text{or} 125, 0.008\right)$
2. $216a$
3. 45

ROUND 5

1. $\frac{p}{4}$
2. $2\sqrt{5}, 4\sqrt{2}$
3. $\left(3\sqrt{2}, -\sqrt{6}\right)$

TEAM

3 pts. 1. 40, 169

3 pts. 2. $\frac{3255}{32768} \approx 0.09933$

4 pts. 3. $18 - 6\sqrt{3}$

Detailed Solutions for Meet 5 GBML 2002

ROUND 1 – Arithmetic

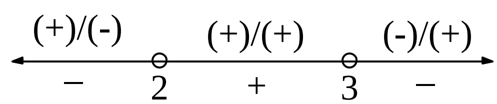
1. Let the numerator = $12x \Rightarrow$ the denominator = $13x \Rightarrow$
$12x+13x=2000 \Rightarrow 25x=2000 \Rightarrow x=80 \Rightarrow 13x=1040$.

2. If the number is 2-digit $\Rightarrow 10t+u=12(t+u) \Rightarrow 2t+11u=0$, which is clearly impossible. If the number is 3-digit $\Rightarrow 100h+10t+u=12(h+t+u) \Rightarrow 88h=2t+11u$ which is true only if $u=8$ and $t=0 \Rightarrow h=1 \Rightarrow$ the number is 108.

3. $2002 = 2\times 7\times 11\times 13$; any whole number with 4 factors is of the form $p_1 \cdot p_2$ or p_1^3 where p_1 and p_2 are primes; since 2002 is the product of 4 distinct primes, there are ${}_4C_2$ = 6 different pairs of primes $\Rightarrow$ there are 6 values for n.

ROUND 2 – Algebra 1

1. $\frac{x+2}{x-2}>5 \Rightarrow \frac{x+2}{x-2}-\frac{5}{1}>0 \Rightarrow \frac{x+2-5x+10}{x-2}>0 \Rightarrow \frac{12-4x}{x-2}>0$; key values for the inequality are 2 and 3 (the numbers that make the numerator and denominator 0). Now section off the number line using 2 and 3 and check each interval. Therefore the solution set is $\{\, x \mid 2<x<3 \}$.



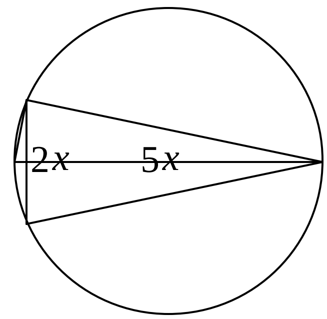
2. Since the points $A(6,-2)$, $B(t+1,-4)$, and $C(t,4)$ form a right angle at A, the slope of $\overline{AB}$ times the slope of $\overline{AC}$ equals $-1 \Rightarrow \left(\frac{-4+2}{t+1-6}\right)\left(\frac{4+2}{t-6}\right)=-1 \Rightarrow \left(\frac{-2}{t-5}\right)\left(\frac{6}{t-6}\right)=-1 \Rightarrow$
$t^2-11t+30=12 \Rightarrow t^2-11t+18=0 \Rightarrow (t-2)(t-9)=0 \Rightarrow t=2,\ 9$.

3. Let x = number of adults, y = number of children, and z = number of senior citizens $\Rightarrow$
$x+y+z=100$ *and* $5.5x+4.5y+4z=465$ *and* $z>y \Rightarrow$
$5.5x+5.5y+5.5z=550$ *and* $5.5x+4.5y+4z=465 \Rightarrow y+1.5z=85$. If $z=y$, then $2.5y=85 \Rightarrow y=z=34$. Since $z>y$, then the smallest value for z would be 36 $\Rightarrow$ $y+54=85 \Rightarrow y=31$ is the largest possible value.

ROUND 3 – Geometry

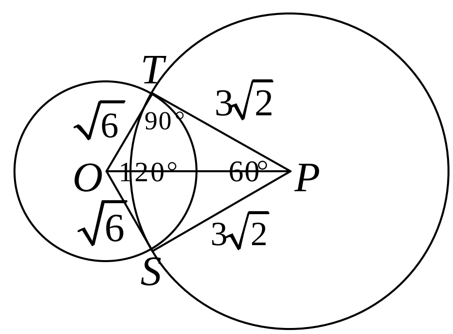
1. The base must be an odd integer. Call its length $2x+1$. The sum of the lengths of the legs must then = $111-(2x+1)=110-2x$. By the triangle inequality theorem, $110-2x>2x+1 \Rightarrow 4x<109 \Rightarrow x<27\frac{1}{4}$. This means x can be any whole number from 0 to 27 $\Rightarrow$ 28 possibilities.

2. Let the length of the diagonals = $5x$ and $2x$ $\Rightarrow$
$\frac{1}{2}(2x)(5x)=4 \Rightarrow x^2=\frac{4}{5} \Rightarrow x=\frac{2}{\sqrt{5}} \Rightarrow 5x=\frac{10}{\sqrt{5}}=2\sqrt{5}$.



Since a circle can be circumscribed around the kite, the angles opposite the longer diagonal are supplementary. Since they are also equal, then they are right and so the diameter of the circle = $2\sqrt{5} \Rightarrow A=p\left(\sqrt{5}\right)^2=5p$.

3. $\angle T$ is right and $\frac{PT}{OT}=\frac{3\sqrt{2}}{\sqrt{6}}=\sqrt{3} \Rightarrow \Delta\ OPT$ is a 30-60-90° triangle $\Rightarrow m\angle SOT=120°$ and $m\angle SPT=60°$



The common area of the two circles is a 120° and 60° segment of circles O and P respectively $\Rightarrow$ Common area

$$=\left(\frac{1}{3}p\left(\sqrt{6}\right)^2-\frac{\left(\sqrt{6}\right)^2\sqrt{3}}{4}\right)+\left(\frac{1}{6}p\left(3\sqrt{2}\right)^2-\frac{\left(3\sqrt{2}\right)^2\sqrt{3}}{4}\right)=$$

$$\left(2p-\frac{3\sqrt{3}}{2}\right)+\left(3p-\frac{9\sqrt{3}}{2}\right)=5p-6\sqrt{3}.$$

ROUND 4 – Algebra 2

1. $\log_{25}x=18\log_{x^4}5 \Rightarrow \frac{\log x}{\log 25}=\frac{18\log 5}{\log x^4} \Rightarrow \frac{\log x}{2\log 5}=\frac{18\log 5}{4\log x} \Rightarrow (\log x)^2=9(\log 5)^2 \Rightarrow$
$\log x=\pm 3\log 5 \Rightarrow \log x=\log 5^{\pm 3} \Rightarrow x=5^{\pm 3}=125, \frac{1}{125}$.

2. Let the four terms be $a, ar, ar^2, ar^3 \Rightarrow \frac{a+ar+ar^2+ar^3}{a+ar}=37 \Rightarrow \frac{1+r+r^2+r^3}{1+r}=37 \Rightarrow$
$\frac{(1+r)(1+r^2)}{1+r}=37 \Rightarrow 1+r^2=37 \Rightarrow r=6 \Rightarrow$ fourth term is $216a$.

3. $(i)\ 4+k=(a-d)^2$, $(ii)\ 124+k=a^2$, and $(\text{iii})\ 316+k=(a+d)^2 \Rightarrow$
$(ii)-(i):120=2ad-d^2$ and $(iii)-(ii):192=2ad+d^2$. Subtracting these equations:
$2d^2=72 \Rightarrow d=\pm 6 \Rightarrow \pm 12a+36=192 \Rightarrow a=\pm 13 \Rightarrow 124+k=169 \Rightarrow k=45$.

ROUND 5 – Precalculus

1. Since $x = \text{Arctan}\left(\frac{1}{2}\right) + \text{Arctan}\left(\frac{1}{3}\right) \Rightarrow \tan x = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = 1 \Rightarrow x = \frac{p}{4}$.

2. $\frac{1}{2}(1)(5)\sin A = 2 \Rightarrow \sin A = \frac{4}{5} \Rightarrow \cos A = \pm\frac{3}{5}$. By the Law of cosines,
$BC^2 = 1^2 + 5^2 - 2(1)(5)\left(\pm\frac{3}{5}\right) \Rightarrow BC^2 = 26 \pm 6 = 20, 32 \Rightarrow BC = 2\sqrt{5}, 4\sqrt{2}$.

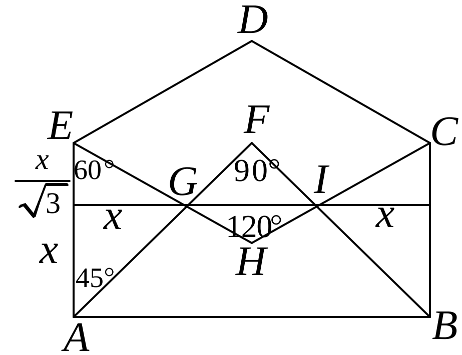
3. The terminal side of the –30° angle has slope = $\tan(-30°) = -\frac{\sqrt{3}}{3} \Rightarrow$ equation of the terminal side is $y = -\frac{\sqrt{3}}{3}x,\ x > 0$; for the hyperbola, $a = \sqrt{6},\ c = 3 \Rightarrow b^2 = 3^2 - \sqrt{6}^2 = 3$.
The transverse axis is the x axis $\Rightarrow$ equation of the hyperbola is $\frac{x^2}{6} - \frac{y^2}{3} = 1 \Rightarrow$
$\frac{x^2}{6} - \frac{\left(-\frac{\sqrt{3}}{3}x\right)^2}{3} = 1 \Rightarrow \frac{x^2}{6} - \frac{x^2}{9} = 1 \Rightarrow x^2 = 18 \Rightarrow x = 3\sqrt{2} \Rightarrow y = -\frac{\sqrt{3}}{3}\left(3\sqrt{2}\right) = -\sqrt{6} \Rightarrow$
the point of intersection is $\left(3\sqrt{2}, -\sqrt{6}\right)$.

TEAM ROUND

1. Let x = number picked $\Rightarrow 4\left(3\left(2(x+1)+1\right)+1\right) = 4(6x+10) = 8(3x+5)$ = perfect cube
$\Rightarrow 3x+5$ is a perfect cube, call it y^3. $y^3 = 3x+5 \equiv 2\text{mod}3 \Rightarrow$ since $(2\text{mod}3)^3 = 8\text{mod}3$
$\equiv 2\text{mod}3 \Rightarrow y = 2\text{mod}3$. The 2 values for y are 5, 8 $\Rightarrow 3x+5 = 5^3$ or $8^3 \Rightarrow x = 40, 169$.

2. $P(T) = \left(\frac{{}_4C_0}{2^4}\right)^2 + \left(\frac{{}_4C_1}{2^4}\right)^2 + \left(\frac{{}_4C_2}{2^4}\right)^2 + \left(\frac{{}_4C_3}{2^4}\right)^2 + \left(\frac{{}_4C_4}{2^4}\right)^2 = \frac{35}{128} \Rightarrow$
$P(W) = \left(1 - \frac{35}{128}\right) \div 2 = \frac{93}{256} \Rightarrow P(TW) = \left(\frac{35}{128}\right)\left(\frac{93}{256}\right) = \frac{3255}{32768} \approx 0.09933$

3. Draw the line through G and I and let x = altitude of ΔAGE from G. Because of the special right triangles,
$AE = 6 = x + \frac{x}{\sqrt{3}} \Rightarrow x = \frac{6}{1+\frac{1}{\sqrt{3}}} = \frac{3}{2}(6)\left(1 - \frac{\sqrt{3}}{3}\right) =$
$9 - 3\sqrt{3}$; $GI = 12 - 2x = 6\sqrt{3} - 6$; area of $FGHI$ = area of ΔGIF + area of ΔGIH =
$\frac{1}{4}\left(6\sqrt{3}-6\right)^2 + \frac{1}{4}\left(6\sqrt{3}-6\right)^2\left(\frac{\sqrt{3}}{3}\right) = 18 - 6\sqrt{3}$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2003

ROUND 1 – Arithmetic
Problem submitted by Chelmsford.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The 3-digit base 6 number $5d4_6$ is a perfect square. Find all possible values for the digit d.

2. It is true that 2003 is prime. To prove this, you could divide 2003 by all the prime numbers from 2 to p. Find the smallest possible value for p.

3. Consider 9 dates in a 3×3 square of numbers taken from a calendar of an arbitrary month shown on the right. If the sum of the five numbers in the positions marked with an **X** is 105, what is the sum of the numbers in the other four positions?

X		**X**
	X	
X		**X**

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2003

ROUND 2 – Algebra 1
Problem submitted by Chelmsford.

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given five positive numbers m, n, x, y, and z such that $x^3 + y^3 = z^2$, $y = mx$, and $z = nx$, solve for x in terms of m and n.

2. Find the value of $\sqrt{\left(1-2x^{-1}\right)\left(1+2x^{-1}+4x^{-2}\right)}$ if $x = -\frac{1}{\sqrt[3]{3}}$.

3. In a game a red chip is worth 10 points, a blue chip is worth 3 points, and a white chip is worth 1 point. A player notes the value of his red chips is worth 40 points more than the total value of his blue and white chips. The number of red chips is one-third the total number of blue and white chips. All his chips are worth 240 points. How many chips does the player have?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2003

ROUND 3 – Geometry

1. ____________________

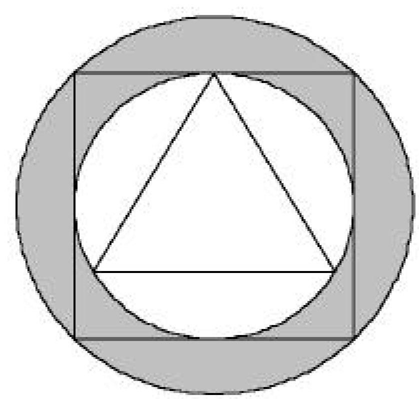
2. ____________________

3. ____________________

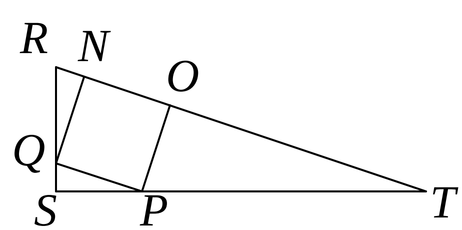
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The three sides of a triangle has integral lengths of $x+8$, 20, and $3x$. How many different triangles have this property?

2. In the diagram on the right, a square is inscribed in a circle, a second circle is inscribed in the square, and an equilateral triangle of area $9\sqrt{3}$ is inscribed in the second circle. Find the shaded area.



3. Given ΔRST with $\angle S$ right, $RS = 1$ and $ST = 3$, square $NOPQ$ is constructed with vertices on the triangle as shown on the diagram on the right. Find the area square $NOPQ$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2003

ROUND 4 – Algebra 2
Problem submitted by Chelmsford.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Solve the following equation for *x*: $\log_4 49 - \log_2 x = \log_8 125$

2. Let S and *P* denote the sum and the product respectively of the roots for *x* of the equation, $x^2 + a^2 + ax = x + \frac{25}{36}$. For what value or values of a does $S - P = \frac{3}{2}$?

3. The first, second, and sixth term of an arithmetic sequence form three consecutive, unequal terms of a geometric sequence. Given the sum of the first 3 terms of the arithmetic sequence is 18, find the fourth term of the geometric sequence.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2003

ROUND 5 – Precalculus

1. $(\_\_\_\_,\_\_\_\_)$

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Conic C, $\left\{(x, y) \middle| x^2 - y^2 = 1\right\}$, contains point P in the first quadrant 5 units from the center of C. Find the coordinates of point P.

2. Given $z^3 = (1+i)\left(\sqrt{3}-i\right)$, solve for z in the form $r\,cis\,\theta$ with $r > 0$, $0° \le \theta < 360°$, and the value for r is expressed in simplified radical form. Note that $r\,cis\,\theta = r(\cos\theta + i\sin\theta)$ where $i = \sqrt{-1}$.

3. Given $0° \le \theta < 360°$ and $\cos\theta \cdot \cos 350° + \sin\theta \cdot \cos 100° = \frac{\sqrt{3}}{2}$, solve for θ.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2003

TEAM ROUND: Time limit: 12 minutes

3 pts. 1. ____________________

3 pts. 2. ____________________

4 pts. 3. ____________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND except for the TI-89 Calculator or <u>any</u> calculator with symbolic operation capabilities, which are not allowed on the Team Round

1. If you divide one more than a cube of an integer by two more than the integer, you have an integer quotient. Find the largest possible integer quotient.

2. Line L_1 has a slope of 2 and contains point $(0,b)$, $b > 0$. Line L_2 is parallel to L_1 and has a y-intercept 3 more than L_1. If the area of the trapezoid formed by the two lines and the coordinate axes is 12, find the value of b.

3. In a poker game with a standard deck of cards (no jokers), the first 3 cards dealt to one player are a pair of 5's (same rank) and a King. What is the probability that the next two cards dealt to the player will improve the value of the "poker hand"? Write the result as a reduced rational number.

   Note: In poker, the types of 5 card hands with more value than 1 pair are (i) 2 pairs, (ii) 3 of the same rank, (iii) 3 of one rank and 2 of another, or (iv) 4 of the same rank. (No other poker hand with value more than 1 pair is possible under the given conditions.)

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2003

ANSWER SHEET:

ROUND 1

1. 2

2. 43

3. 84

ROUND 2

1. $x = \dfrac{n^2}{1+m^3}$

2. 5

3. 56

ROUND 3

1. 10

2. $12p$

3. $\dfrac{90}{169}$

ROUND 4

1. $\dfrac{7}{5}$ $\left(1\tfrac{2}{5} \text{ or } 1.4\right)$

2. $-\dfrac{7}{6}, \dfrac{1}{6}$

3. 96

ROUND 5

1. $\left(\sqrt{13}, 2\sqrt{3}\right)$

2. $\sqrt{2}\,cis\,5°, \sqrt{2}\,cis\,125°, \sqrt{2}\,cis\,245°$

3. 20°, 320°

TEAM

3 pts. 1. 104

3 pts. 2. $\dfrac{13}{2}$ $\left(6\tfrac{1}{2} \text{ or } 6.5\right)$

4 pts. 3. $\dfrac{37}{147}$

Detailed Solutions for Meet 5 GBML 2003

ROUND 1 – Arithmetic

1. $5d4_6 = 5 \times 36 + 6d + 4 = 184 + 6d$; the first perfect square greater than 184 is 196 $\Rightarrow$ $184 + 6d = 196 \Rightarrow d = 2$; no other perfect square greater than 196 will produce an integer value for d less than 6 $\Rightarrow$ d = 2 only.

2. To find the smallest possible value for p, if q is the next prime number, then $p^2 < 2003$ and $q^2 > 2003$. $43^2 = 1849$ and $47^2 = 2209 \Rightarrow p = 43$.

3. If n = date of the upper left hand entry, then the other entries are as shown on the right $\Rightarrow$ $5n + 40 = 105 \Rightarrow$ $n = 13$; the other four entries add to $4n + 32 = 52 + 32 = 84$.

n	$n + 1$	$n + 2$
$n + 7$	$n + 8$	$n + 9$
$n + 14$	$n + 15$	$n + 16$

ROUND 2 – Algebra 1

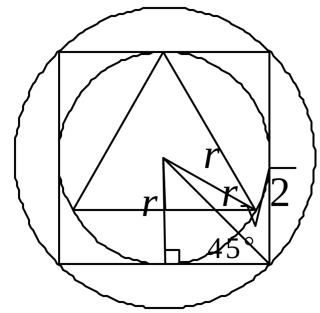
1. $x^3 + y^3 = z^2$, $y = mx$, and $z = nx \Rightarrow x^3 + (mx)^3 = (nx)^2 \Rightarrow x^3 + m^3x^3 = n^2x^2 \Rightarrow$
$x^3(1 + m^3) = n^2x^2 \Rightarrow$ since $x \neq 0$, then $x(1 + m^3) = n^2 \Rightarrow x = \dfrac{n^2}{1 + m^3}$.

2. $$\sqrt{(1 - 2x^{-1})(1 + 2x^{-1} + 4x^{-2})} = \sqrt{1 - 8x^{-3}} = \sqrt{1 - 8\left(-\frac{1}{\sqrt[3]{3}}\right)^{-3}} = \sqrt{1 - 8(-3)} = \sqrt{25} = 5$$

3. Let r = number of red chips, w = number of white chips, and b = number of blue chips $\Rightarrow$ $10r = 3b + w + 40$, $r = \dfrac{1}{3}(b + w)$, and $10r + 3b + w = 240 \Rightarrow$ adding the first and last equations, $20r = 280 \Rightarrow r = 14 \Rightarrow b + w = 42 \Rightarrow r + b + w = 56$.

ROUND 3 – Geometry

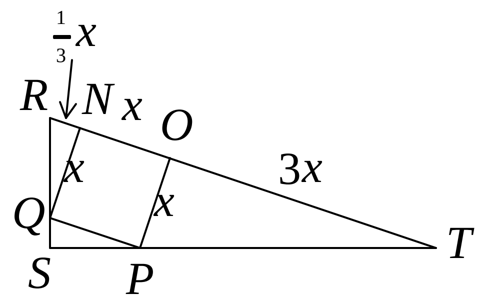
1. By the triangle inequality theorem, $4x+8>20 \Rightarrow x>3$, $x+28>3x \Rightarrow x<14$, $3x+20>x+8 \Rightarrow x>-6$; the intersection of these three inequalities is $3<x<14 \Rightarrow$ $x=4,5,6,...,13$, which imply there are 10 different triangles.



2. Let s = side of equilateral triangle $\Rightarrow \dfrac{s^2\sqrt{3}}{4}=9\sqrt{3} \Rightarrow s^2=36 \Rightarrow$

$s=6 \Rightarrow$ if r = radius of the inner circle, then $r=\dfrac{3}{\sqrt{3}}\cdot 2=2\sqrt{3}$

and if R = radius of the outer circle, then $R=r\sqrt{2}=2\sqrt{6}$;
the shaded area = $p\left(R^2-r^2\right)=p\left(24-12\right)=12p$.



3. $RT=\sqrt{1^2+3^2}=\sqrt{10}$; let side of square = $x \Rightarrow$

by similar triangles, $RN=\dfrac{1}{3}x$ and $OT=3x \Rightarrow$

$\dfrac{1}{3}x+x+3x=\sqrt{10} \Rightarrow \dfrac{13}{3}x=\sqrt{10} \Rightarrow x^2=\dfrac{90}{169}$.

ROUND 4 – Algebra 2

1. $\log_4 49-\log_2 x=\log_8 125 \Rightarrow \log_4 49-\log_8 125=\log_2 x \Rightarrow \log_{2^2} 7^2-\log_{2^3} 5^3=\log_2 x \Rightarrow$
$\log_2 7-\log_2 5=\log_2 x \Rightarrow \log_2\left(\dfrac{7}{5}\right)=\log_2 x \Rightarrow x=\dfrac{7}{5}$.

2. $x^2+a^2+ax=x+\dfrac{25}{36} \Rightarrow x^2+ax-x+a^2-\dfrac{25}{36}=0 \Rightarrow x^2+x(a-1)+\left(a^2-\dfrac{25}{36}\right)=0 \Rightarrow$

$S=1-a$ and $P=a^2-\dfrac{25}{36} \Rightarrow (1-a)-\left(a^2-\dfrac{25}{36}\right)=\dfrac{3}{2} \Rightarrow$

$a^2+a-\dfrac{7}{36}=0 \Rightarrow 36a^2+36a-7=0 \Rightarrow (6a+7)(6a-1)=0 \Rightarrow a=-\dfrac{7}{6},\dfrac{1}{6}$

3. Let a = first term and d = difference between terms:
$a(a+5d)=(a+d)^2 \Rightarrow a^2+5ad=a^2+2ad+d^2 \Rightarrow 5ad=2ad+d^2 \Rightarrow 3ad=d^2 \Rightarrow$

$d=3a\ (d\neq 0)$; $a+4a+7a=18 \Rightarrow a=\dfrac{3}{2}$; $r=\dfrac{4a}{a}=4 \Rightarrow$

fourth term of the geometric sequence = $\dfrac{3}{2}\cdot 4^3=96$.

ROUND 5 – Precalculus

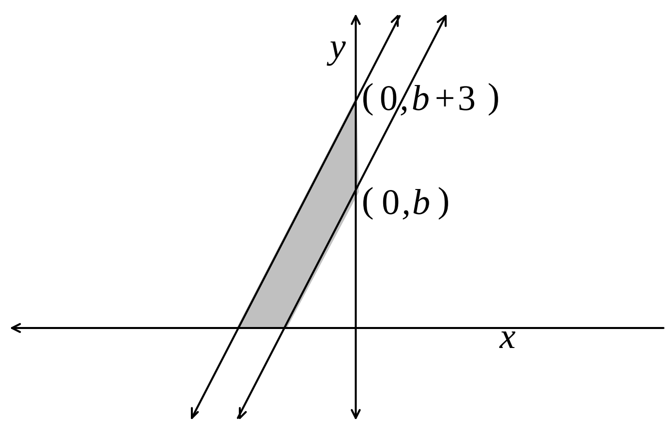
1. Let the coordinates of $P = (x, y): x^2 + y^2 = 25$ and $x^2 - y^2 = 1 \Rightarrow$
$2x^2 = 26 \Rightarrow x^2 = 13 \Rightarrow y^2 = 12 \Rightarrow \; P = \left(\sqrt{13}, 2\sqrt{3}\right)$ (P is in quadrant I)

2. $1 + i = \sqrt{2}\, cis\, 45°$ and $\sqrt{3} - i = 2\, cis(-30°) \Rightarrow z^3 = 2\sqrt{2}\, cis\, 15° \Rightarrow$
$z = \left(2\sqrt{2}\right)^{1/3} cis\left(\frac{15°}{3} + \frac{360°}{3}k\right)\Big| k = 0,1,2 \Rightarrow z = \sqrt{2}\, cis\, 5°, \sqrt{2}\, cis 125°, \sqrt{2}\, cis\, 245°$

3. $\cos q \cdot \cos 350° + \sin q \cdot \cos 100° = \frac{\sqrt{3}}{2} \Rightarrow \cos q \cdot \cos 10° - \sin q \cdot \sin 10° = \frac{\sqrt{3}}{2} \Rightarrow$
$\cos(q + 10°) = \cos 30°$ or $\cos 330° \Rightarrow q = 20°, 320°$.

TEAM ROUND

1. Let x = integer: $x + 2\overline{)x^3 + 1}$ done synthetically: $\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 1 \\ & 1 & -2 & 4 & -7 \end{array}$ $\Rightarrow$ result is
$x^2 - 2x + 4 - \frac{7}{x+2}$; to maximize the quotient
let $x = -9 \Rightarrow$ result $= 81 + 18 + 4 + 1 = 104$.



2. The equations of the two lines are:
$y = 2x + b,\; y = 2x + b + 3$; the x-intercepts
of these lines are: $\left(-\frac{b}{2}, 0\right)$ and $\left(-\frac{b+3}{2}, 0\right)$;
the shaded area = $\frac{(b+3)^2}{4} - \frac{b^2}{4} = 12 \Rightarrow 6b + 9 = 48 \Rightarrow b = \frac{13}{2}$

3. The number of elements in the sample space = ${}_{49}C_2$; successful events are being dealt a pair other than a 5 or King: $11 \cdot {}_4C_2 = 66$ ways; dealt 3rd 5 and a card other than a 5 $2 \cdot 47 = 94$ ways; dealt a King and a card other a King or 5: $3 \cdot 44 = 132$ ways; dealt 2 King's = 3 ways; dealt 2 5's = 1 way. Note all these sets of hands are disjoint.
Therefore the probability of the hand improving in value =
$\frac{66 + 94 + 132 + 3 + 1}{{}_{49}C_2} = \frac{37}{147}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2004

ROUND 1 – Arithmetic
Problems submitted by Maimonides.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. What is the sum of all the digits in the decimal representation of $10^{99} - 99$?

2. Find the value of $\sqrt{0.\overline{222}_{\text{nine}}} + \sqrt{0.\overline{444}_{\text{ten}}}$ as a base ten numeral.

3. How many four-digit whole numbers contain 12 as consecutive digits in that order?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2004

ROUND 2 – Algebra 1
Problem submitted by Belmont.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Line ℓ contains point $P(3,1)$ and bisects the segment of the line $5x-2y-20=0$ determined by the coordinate axes. Find the slope of line ℓ.

2. Find all solutions to the equation $|x+2|=2|x-2|$.

3. Al, Bill, and Cassie run a 6 mile race. Cassie beats Al by 27 minutes, Bill beats Al by 24 minutes, and Cassie's average speed is 3 miles per hour faster than Al's. Find the number of miles per hour in Bill's average speed.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2004

ROUND 3 – Geometry

Problems submitted by Belmont and Maimonides.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A sphere is circumscribed about a cube with side of length 2. Find the volume of the sphere.

2. A square and an equilateral triangle have equal perimeters. Circle A is circumscribed about the square and circle B is circumscribed about the equilateral triangle. Find the ratio of the areas of circle A to circle B.

3. A triangle is inscribed in a semicircle with radius 2.5 so that one side of the triangle is the diameter of the semicircle. If the perimeter of this triangle is 11, find its area.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2004

ROUND 4 – Algebra 2
Problem submitted by Maimonides.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the following product: $(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)...(\log_{127} 128)$.

2. Given x is an element of the complex numbers and 3 is a solution to the equation $x^3 - 7x^2 + ax - 15 = 0$, find the remaining solutions to the equation.

3. Given the recursive sequence $a_n = a_{n-1} - a_{n-2}$, $a_1 = 2$ and $a_3 = 3$. Find a_{2004}.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2004

ROUND 5 – Precalculus

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Simplify $\left(\dfrac{\sqrt{3}+i}{\sqrt[3]{2}}\right)^9$.

2. If $\sin x = -\dfrac{\sqrt{7}}{4}$ and $\sec x > 0$, determine the value of $\tan 2x$.

3. Hyperbola H has center $C(2,-3)$, vertex $V(6,-3)$ and focus $F(7,-3)$. H also contains point P in the first quadrant with y-coordinate $2\sqrt{10}-3$. Find the x-coordinate of P.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2004

TEAM ROUND: Time limit: 12 minutes

Problem submitted by Belmont

3 pts. 1. ___________________

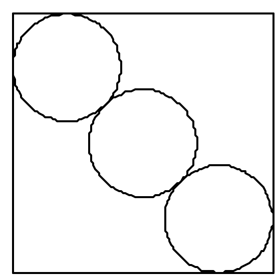
3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND except for the TI-89 Calculator or <u>any</u> calculator with symbolic operation capabilities, which are not allowed on the Team Round

1. Given the ordered pair (x, y) of real numbers satisfying the relation $4y^2 + 4xy + x + 6 = 0$, find the domain of this relation.

2. Given a square of side 4, three congruent circles are drawn two of which are tangent to the sides of the square and all three are externally tangent in pairs as shown. Find the radius of one of the circles.



3. The digits from 1 to 9 are written on nine index cards, one digit per card. Three cards are chosen at random without replacement. Given that exactly one of the digits chosen is divisible by 3, what is the probability that the sum of the three digits on the three cards is at least 18?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2004

ANSWER KEY:

ROUND 1

1. 874
2. $\frac{7}{6}$
3. 279

ROUND 2

1. 6
2. $\frac{2}{3}, 6$
3. $\frac{15}{2}$ $(7.5 \text{ or } 7\tfrac{1}{2})$

ROUND 3

1. $4\sqrt{3}p$
2. $27:32$
3. $\frac{11}{4}$ $(3.75 \text{ or } 3\tfrac{3}{4})$

ROUND 4

1. 7
2. $2 \pm i$
3. -3

ROUND 5

1. $-64i$ $(0 - 64i)$
2. $-3\sqrt{7}$
3. $\frac{34}{3}$ $(11\tfrac{1}{3})$

TEAM

3 pts. 1. $x \geq 3$ or $x \leq -2$

3 pts. 2. $2\sqrt{2} - 2$

4 pts. 3. $\frac{14}{45}$

Detailed Solutions for Meet 5 GBML 2004

ROUND 1 – Arithmetic

1. $10^{99} - 99 = \underbrace{999....901}_{97\ 9's} \Rightarrow$ sum of the digits $= 97 \times 9 + 1 = 874$

2. $0.\overline{222}_{\text{nine}} = \frac{2}{9} + \frac{2}{9^2} + \frac{2}{9^3} + ... = \frac{2/9}{1 - 1/9} = \frac{1}{4}$; $0.\overline{444}_{\text{ten}} = \frac{4}{9}$ $\sqrt{\frac{1}{4}} + \sqrt{\frac{4}{9}} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$.

3. Case 1: 12 A B of which there are 100 possibilities. Case 2: A 12 B of which there are 90 possibilities. Case 3: A B 12 of which there are 90 possibilities. There is one duplicated number: 1212 $\Rightarrow 100 + 90 + 90 - 1 = 279$ possibilities.

ROUND 2 – Algebra 1

1. The line $5x - 2y = 20$ intersects the axes at points $(4,0)$ and $(0,-10)$. The midpoint between these points is $(2,-5)$. The slope between $(2,-5)$ and $(3,1)$ = $\frac{1+5}{3-2} = 6$.

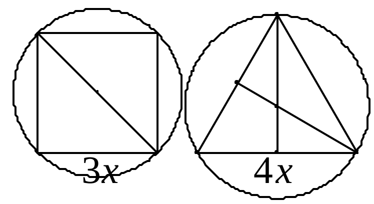
2. The key numbers for this equation are –2 and 2. $x \geq 2 : x + 2 = 2(x-2) \Rightarrow$
$x + 2 = 2x - 4 \Rightarrow x = 6$; $-2 < x < 2 : x + 2 = -2(x-2) \Rightarrow x + 2 = -2x + 4 \Rightarrow x = \frac{2}{3}$;
$x \leq -2 : -(x+2) = -2(x-2) \Rightarrow x = \cancel{6} \Rightarrow$ the solutions are $\frac{2}{3}, 6$

3. Let a = Al's rate in mph, let b = Bill's rate, and let c = Cassie's rate: $\frac{6}{a} - \frac{6}{c} = \frac{9}{20}$,
$\frac{6}{a} - \frac{6}{b} = \frac{2}{5}$, and $c = a + 3 \Rightarrow \frac{6}{a} - \frac{6}{a+3} = \frac{9}{20} \Rightarrow 120(a + 3 - a) = 9a(a+3) \Rightarrow$
$9a^2 + 27a - 360 = 0 \Rightarrow a^2 + 3a - 40 = 0 \Rightarrow (a+8)(a-5) = 0 \Rightarrow a = 5 \Rightarrow$
$\frac{6}{5} - \frac{6}{b} = \frac{2}{5} \Rightarrow \frac{6}{b} = \frac{4}{5} \Rightarrow b = \frac{15}{2}$.

ROUND 3 – Geometry

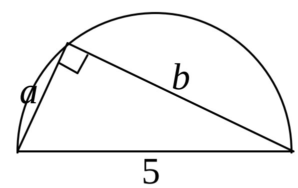
1. The diameter of the sphere = the diagonal of the cube = $2\sqrt{3}$ $\Rightarrow$
$V = \frac{4}{3}p\left(\sqrt{3}\right)^3 = 4\sqrt{3}p$.

2. Since the square and the equilateral triangle have equal perimeters, let side of square = 3*x* and the side of the equilateral triangle = 4*x*. The radius of the circle circumscribing the square $= \frac{3x\sqrt{2}}{2}$; the radius of the circle circumscribing the triangle = $\frac{2}{3}$ of its altitude $= \frac{2}{3}\left(2x\sqrt{3}\right) = \frac{4}{3}x\sqrt{3}$;



the ratio of the areas of the circles = $\left(\frac{\frac{3\sqrt{2}}{2}}{\frac{4\sqrt{3}}{3}}\right)^2 = \frac{9}{2}\cdot\frac{3}{16} = \frac{27}{32}$.

3. The triangle is right with hypotenuse equaling 5. Let *a* and *b* = length of legs $\Rightarrow a^2 + b^2 = 25$. Also $a + b = 6 \Rightarrow$
$a^2 + 2ab + b^2 = 36$. Subtracting the equations: $2ab = 11 \Rightarrow ab = \frac{11}{2}$
and since the area of the triangle = $\frac{1}{2}ab = \frac{1}{2}\cdot\frac{11}{2} = \frac{11}{4}$.



ROUND 4 – Algebra 2

1. $(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)...(\log_{127} 128) =$
$\left(\frac{\log 3}{\log 2}\right)\left(\frac{\log 4}{\log 3}\right)\left(\frac{\log 5}{\log 4}\right)\left(\frac{\log 6}{\log 5}\right)......\left(\frac{\log 128}{\log 127}\right) = \frac{\log 128}{\log 2} = \frac{\log 2^7}{\log 2} = 7$.

2. Using synthetic division: $3 \overline{\Big)\begin{array}{cccc} 1 & -7 & a & -15 \\ 1 & -4 & a-12 & 3a-51 \end{array}}$ $\Rightarrow 3a - 51 = 0 \Rightarrow a = 17 \Rightarrow$
$x^2 - 4x + 5 = 0 \Rightarrow (x-2)^2 = -1 \Rightarrow x - 2 = \pm i \Rightarrow x = 2 \pm i$.

3. $a_3 = a_2 - a_1 \Rightarrow 3 = a_2 - 2 \Rightarrow a_2 = 5$; now generate subsequent terms:
$a_4 = 3 - 5 = -2,\ a_5 = -2 - 3 = -5,\ a_6 = -5 - (-2) = -3,\ a_7 = -3 - (-5) = 2,\ \ a_7 = 2 - (-3) = 5$;
therefore the terms in this recursive sequence repeat every 6 terms and since $2004 \div 6$ has a remainder 0 $\Rightarrow a_{2004} = a_6 = -3$.

ROUND 5 – Precalculus

1. $\left(\frac{\sqrt{3}+i}{\sqrt[3]{2}}\right)^9 = \left(\frac{2\,cis\,30°}{\sqrt[3]{2}}\right)^9 = \frac{2^9\,cis\,270°}{2^3} = -2^6 i = -64i.$

2. Since $\sec x > 0 \Rightarrow \cos x > 0$; $\left(-\frac{\sqrt{7}}{4}\right)^2 + \cos^2 x = 1 \Rightarrow \cos^2 x = \frac{9}{16} \Rightarrow$

$\cos x = \frac{3}{4} \Rightarrow \tan x = -\frac{\sqrt{7}}{4}\cdot\frac{4}{3} = -\frac{\sqrt{7}}{3} \Rightarrow \tan 2x = \frac{2\left(-\sqrt{7}/3\right)}{1-\left(-\sqrt{7}/3\right)^2} = -\frac{2\sqrt{7}}{3}\cdot\frac{9}{2} = -3\sqrt{7}$.

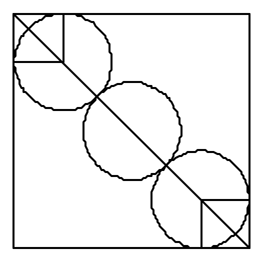
3. For the hyperbola $a = 6-2 = 4$ and $c = 7-2 = 5 \Rightarrow b^2 = 25-16 = 9 \Rightarrow$ equation of the hyperbola is $\frac{(x-2)^2}{16} - \frac{(x+3)^2}{9} = 1 \Rightarrow \frac{(x-2)^2}{16} - \frac{\left(2\sqrt{10}-3+3\right)^2}{9} = 1 \Rightarrow$

$\frac{(x-2)^2}{16} - \frac{40}{9} = 1 \Rightarrow \frac{(x-2)^2}{16} = \frac{49}{9} \Rightarrow \frac{x-2}{4} = \frac{7}{3} \Rightarrow x-2 = \frac{28}{3} \Rightarrow x = \frac{34}{3}$.

TEAM ROUND

1. $4y^2 + 4xy + x + 6 = 0 \Rightarrow y = \frac{-4x \pm \sqrt{16x^2 - 4(4)(x+6)}}{8} = \frac{-4x \pm 4\sqrt{x^2-(x+6)}}{8}$; to find the domain solve the inequality $x^2 - x - 6 \ge 0 \Rightarrow (x-3)(x+2) \ge 0 \Rightarrow x \ge 3$ or $x \le -2$.

2. The diagonal of the square = $4\sqrt{2}$. If r = radius of one circle,



then $4r + 2r\sqrt{2} = 4\sqrt{2} \Rightarrow r = \frac{4\sqrt{2}}{4+2\sqrt{2}} = \frac{2\sqrt{2}}{2+\sqrt{2}} = \frac{2\sqrt{2}\left(2-\sqrt{2}\right)}{\left(2+\sqrt{2}\right)\left(2-\sqrt{2}\right)} =$

$\frac{2\sqrt{2}\left(2-\sqrt{2}\right)}{2} = 2\sqrt{2}-2$.

3. The sample space has ${}_3C_1 \times {}_6C_2 = 45$ elements. If 3 is on one card $\Rightarrow (8,7)$ is the only possible combination for the other two cards. If 6 is on one card $\Rightarrow (8,7), (8,5), (8,4)$, and $(7,5)$ are the only possible combinations for the other two cards.
If 9 is on one card $\Rightarrow (8,7), (8,5), (8,4), (8,2), (8,1), (7,5), (7,4), (7,2)$, and $(5,4)$ are the only possible combinations for the other two cards. Therefore, there are 14 successful events and the probability = $\frac{14}{45}$.